Stochastic Permanent Income Model and Government Fiscal Policy

Honours Intermediate Macro

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Stochastic Permanent Income

Basic Setup

Linear State Space + Normal Shock:

Let

$$
\begin{aligned}
x_{t+1} &= Ax_t + Cw_{t+1} \\
y_t &= G \cdot x_t
\end{aligned}
$$

where A is $n \times n$ matrix, x is $n \times 1$ vector, C is $n \times m$ matrix, $w_{t+1} \sim N(0, I_{m\times m})$ are i.i.d. normal shocks; G is $1 \times n$ vector, y_t is a scalar representing labor income.

Consumer's Budget Constraint (assuming $\beta R = 1$):

$$
F_{t+1} = \underbrace{\frac{1}{\beta}}_{\substack{\text{gross}\\\text{interest}\\\text{rate}}} (\underbrace{F_t}_{\substack{\text{Financial}\\\text{wealth}}} + y_t - c_t) \tag{1}
$$

Recall if $\{y_t\}$ is deterministic, and $R = 1/\beta$, then for any strictly concave $u(c)$ they achieved perfect consumption smoothing:

$$
c_t = (1-\beta)\Big(\underbrace{F_t}_{\substack{\text{Financial}\\\text{wealth}}} + \underbrace{\sum_{j=0}^{\infty} \beta^j y_{t+j}}_{\substack{\text{PDV of}\\\text{human}\\\text{wealth}}}\Big)
$$

If y_t is stochastic, can we just replace the above equation with expected value?

$$c_t = (1-\beta)\Big(F_t + \underbrace{\mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j y_{t+j}\right]}_{\substack{\text{expected PDV}\\ \text{of human wealth}\\ \text{with information at}\\ \text{time } t}}\Big) \tag{2}$$

Note: If $u'(c)$ is not linear, then this is only an approximation.

Combine Equation 1 and Equation 2:

$$\begin{aligned} F_{t+1} &= \frac{1}{\beta}\left[F_t + y_t - (1-\beta)\left(F_t + \mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j y_{t+j}\right]\right)\right] \\ &= \frac{1}{\beta}\left[\beta F_t + y_t - (1-\beta)\mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j y_{t+j}\right]\right] \\ \Rightarrow F_{t+1} - F_t &= \frac{1}{\beta}\left[y_t - (1-\beta)\mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j y_{t+j}\right]\right] \end{aligned} \tag{3}$$

That is, agents add the difference between y_t and permanent income. Now use Equation 2 at t and $t+1$:

$$\begin{aligned} c_{t+1} &= (1-\beta)\left[F_{t+1} + \mathbb{E}_{t+1}\left[\sum_{j=0}^{\infty}\beta^j y_{t+j+1}\right]\right] \\ c_t &= (1-\beta)\left[F_t + \mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j y_{t+j}\right]\right] \end{aligned}$$

$$\Rightarrow c_{t+1} - c_t = (1-\beta)(F_{t+1} - F_t) + (1-\beta)\left[\mathbb{E}_{t+1}\left[\sum_{j=0}^{\infty}\beta^j y_{t+j+1}\right] - \mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j y_{t+j}\right]\right]$$

Use Equation 3 to find (after many steps):

$$\boxed{c_{t+1} - c_t = (1-\beta)\sum_{j=0}^{\infty}\beta^j\Big(\underbrace{\mathbb{E}_{t+1}(y_{t+j+1})}_{\substack{\text{Forecast of } t+1, t+2, \ldots\\ \text{with time } t+1\\ \text{information}}} - \underbrace{\mathbb{E}_t(y_{t+j+1})}_{\substack{\text{With time } t\\ \text{information}}}\Big)}$$

- Consumption only changes due to "surprise" of new information changing expected value
- Only **unanticipated** changes in $y_{t+j}, \ldots$ or other information which changes forecasts
- Could be unanticipated changes in government policy or shock realizations

Finally, for a shock between $t \to t+1$ with our linear state space model:

$$\begin{aligned} c_{t+1} - c_t &= (1-\beta)\left[\sum_{j=0}^{\infty} \beta^j \left(\mathbb{E}_{t+1}(y_{t+j+1}) - \mathbb{E}_t(y_{t+j+1})\right)\right] \\ &= (1-\beta)\left[G(I-\beta A)^{-1} x_{t+1} - G(I-\beta A)^{-1} A x_t\right] \\ &= (1-\beta) G (I-\beta A)^{-1}\left[\underbrace{A x_t + C w_{t+1}}_{x_{t+1}} - A x_t\right] \end{aligned}$$

$$\boxed{c_{t+1} - c_t = \underbrace{(1-\beta)}_{\text{Propensity to consume}} \underbrace{G(I-\beta A)^{-1} \cdot C w_{t+1}}_{\text{PDV of impulse response to a shock to } x_{t+1}}}$$

That is, the PDV of changes to forecasts from the realized shock.

Special Case of Quadratic Preferences

Recall Euler equation for Permanent Income Model:

$$u'(c_t) = \beta(1+r) u'(c_{t+1}), \quad \forall t = 0, \ldots, T-1$$

If stochastic consumption and $\beta = \frac{1}{1+r}$, just replace with expectation?

$$\underbrace{u'(c_t)}_{\text{Marginal utility this period}} = \underbrace{\mathbb{E}_t\left[u'(c_{t+1})\right]}_{\text{Expectation of marginal utility next period}}$$

Let $u(c) = \frac{a_1}{2} c^2 + a_2 c + a_3 \Rightarrow u'(c) = a_1 c + a_2$.

In the Euler equation:

$$a_1 c_t + a_2 = \mathbb{E}_t(a_1 c_{t+1} + a_2) \implies c_t = \mathbb{E}_t(c_{t+1})$$

That is, the Euler equation implying perfect consumption smoothing with a deterministic process translates to consumption being a **martingale** if stochastic!

Notes:

- In general, $\mathbb{E}_t(u(c)) \neq u(\mathbb{E}_t(c))$
- Then, we can use the linear-stochastic state space model for forecasting $\mathbb{E}_t(c_{t+1})$
- Due to linearity, it simply forecasts the mean
- This is a general result called **Certainty Equivalence** of optimizing a quadratic objective subject to a linear-Gaussian state space model
- The decision is identical in a model with or without the uncertainty
- However, the realized sequence contingent on the shocks, and utility, are not the same

Certainty Equivalence, Risk, and Prudence

A clean way to see what is special about quadratic utility is to start from the stochastic Euler equation (risk-free return, interior solution). Assume $\beta R = 1$.

$$\boxed{u'(c_t) = \mathbb{E}_t\left[u'(c_{t+1})\right]}$$

- If $u'(\cdot)$ is **linear** (equivalently $u'''(\cdot) = 0$, i.e. u is quadratic), then

  $$\mathbb{E}_t\left[u'(c_{t+1})\right] = u'\left(\mathbb{E}_t[c_{t+1}]\right) \quad \Rightarrow \quad c_t = \mathbb{E}_t[c_{t+1}],$$

  so the **conditional variance** of c_{t+1} does not enter the consumption choice (though it still affects realized utility).

- If $u'''(c) > 0$, then $u'(\cdot)$ is **convex**. By Jensen,

  $$\mathbb{E}_t\left[u'(c_{t+1})\right] \geq u'\left(\mathbb{E}_t[c_{t+1}]\right),$$

  with strict inequality when there is risk. Since u' is decreasing, holding the conditional mean fixed this pushes the Euler equation toward **lower** c_t (higher saving): the precautionary saving motive.

To quantify the effect, take a Taylor expansion of $u'(c_{t+1})$ around c_t and define $\Delta c_{t+1} := c_{t+1} - c_t$:

$$u'(c_{t+1}) \approx u'(c_t) + u''(c_t)\Delta c_{t+1} + \frac{1}{2}u'''(c_t)\Delta c_{t+1}^2.$$

Taking conditional expectations and substituting into the Euler equation gives

$$0 \approx u''(c_t)\,\mathbb{E}_t[\Delta c_{t+1}] + \frac{1}{2}u'''(c_t)\,\mathbb{E}_t[\Delta c_{t+1}^2].$$

Note that $\mathbb{E}_t[\Delta c_{t+1}^2] = \mathrm{Var}_t(\Delta c_{t+1}) + \big(\mathbb{E}_t[\Delta c_{t+1}]\big)^2$. Ignoring the small $(\mathbb{E}_t[\Delta c_{t+1}])^2$ term (a higher-order term in this approximation), we obtain

Define the coefficient of **relative prudence** (appropriate for homothetic preferences):

$$P_R(c) \equiv -\frac{c\,u'''(c)}{u''(c)}.$$

Divide both sides of the approximation by c_t and rewrite the variance in terms of relative consumption growth:

$$\frac{\mathbb{E}_t[\Delta c_{t+1}]}{c_t} \approx \frac{1}{2} P_R(c_t)\ \mathrm{Var}_t\left(\frac{\Delta c_{t+1}}{c_t}\right).$$

Since $u'' < 0$ for concave utility, if $u'''(c) > 0$ then $P_R(c) > 0$. Higher relative risk therefore raises expected consumption growth, implying lower c_t today and more saving (**precautionary saving**).

For quadratic utility $u''' = 0$ (so $P_R = 0$), the approximation reduces to

$$\mathbb{E}_t[\Delta c_{t+1}] = 0,$$

independent of risk: the classic certainty-equivalence intuition.

Example (log utility): If $u(c) = \log c$, then

$$P_R(c) = -\frac{c \cdot (2/c^3)}{-1/c^2} = 2.$$

The relative form becomes

$$\mathbb{E}_t\left[\frac{\Delta c_{t+1}}{c_t}\right] \approx \mathrm{Var}_t\left(\frac{\Delta c_{t+1}}{c_t}\right).$$

Intuitively, the consumer wants to save more today (lower c_t so higher c_{t+1}/c_t) when there is more risk to future consumption growth.

Examples

Pre-announced Tax Cut

This will use a shock to knowledge about deterministic income processes, rather than a constant stream of shocks to income.

Setup:

- Government announces at $t = 0$ that at $t = 1$ it will borrow α from international markets at interest rate $(1 + r)$ per period and give it to each consumer
- They also announce that to eventually balance the budget, they will pay it back at $t = 2$ for simplicity by increasing taxation that period
- Assume consumers had deterministic y_{t+j}. What happens to consumption?

Using our result:

$$c_{t+1} - c_t = (1 - \beta) \sum_{j=0}^{\infty} \beta^j \left[\mathbb{E}_{t+1}(y_{t+j+1}) - \mathbb{E}_t(y_{t+j+1}) \right]$$

Define: $\{\hat{y}_{t+1}\}_{j=0}^{\infty} = \{y_t, \underbrace{y_{t+1} + \alpha, y_{t+2} - \alpha(1 + r)}_{\text{Only difference}}, y_{t+3}, \ldots, y_{t+j} \ldots\}$

- Note that from t to $t + 1$, the agent has the news that $\{y_{t+j}\} \to \{\hat{y}_{t+j}\}$
- This is a change in expectations:

$$\begin{aligned} c_1 - c_0 &= (1 - \beta) \sum_{j=0}^{\infty} \beta^j \left[\mathbb{E}_1(y_{j+1}) - \mathbb{E}_0(y_{j+1}) \right] = (1 - \beta) \sum_{j=0}^{\infty} \beta^j (\hat{y}_{j+1} - y_{j+1}) \\ &= (1 - \beta) \sum_{j=0}^{\infty} \beta^j (y_{j+1} - y_{j+1}) + (1 - \beta) \left[\alpha - \beta(1 + r)\alpha \right] \end{aligned}$$

Result: If $\beta = \frac{1}{1+r}$, then $c_1 - c_0 = 0$.

That is, the tax cut has **no effect** because of the **anticipated rise** in taxes. Later, we will investigate cases why $\beta = \frac{1}{1+r}$ comes out of general equilibrium.

Timing of Tax Cuts

Setup:

- Between time 0 and 1, government announces that it will cut taxes to give α to each individual at a deterministic time $T \geq 1$
- Assume they do not need to pay it back and taxes will not rise to compensate
- What happens to consumption at time $\{0, \ldots, T, T+1, \ldots\}$?
- Assume y_{t+j+1} are deterministic

Solve:

$$
\begin{aligned}
c_1 - c_0 &= (1-\beta)\sum_{j=0}^{\infty} \beta^j \left[\mathbb{E}_1(y_{j+1}) - \mathbb{E}_0(y_{j+1})\right] \\
&= (1-\beta)\sum_{j=0}^{\infty} \beta^j \left[y_{j+1} - y_{j+1}\right] + (1-\beta)\cdot\beta^{T-1}\cdot\alpha \\
&= \underbrace{(1-\beta)}_{\text{MPC out of wealth}} \quad \underbrace{\beta^{T-1}\cdot\alpha}_{\substack{\text{Change in} \\ \text{permanent income}}}
\end{aligned}
$$

For $t \geq 1$:

$$
\mathbb{E}_{t+1}(y_{t+j+1}) = \mathbb{E}_t(y_{t+j+1}) \implies c_{t+1} - c_t = 0, \quad \forall t \geq 1
$$

That is:

- Changes only happen at announcement, not at tax cut time T
- A similar approach with stochastic income would yield the same result

Variation: The only reason that T enters the above is that the PDV of the α delivery is discounted for T periods. If instead, the government announces they will set aside α, put it in the bank at R interest, and then deliver the α with interest at time T. In that case, interest compounds for $T-1$ periods, which means that

$$
c_1 - c_0 = (1-\beta)\beta^{T-1}\left(R^{T-1}\alpha\right) = (1-\beta)\alpha
$$

That is, the tax break (no matter when it is actually implemented) adds α to the PDV of lifetime earnings.

Example from Friedman-Muth

Setup:

$$y_t = z_t + u_t$$
$$z_{t+1} = z_t + \sigma_1 w_{1t+1}$$
$$u_{t+1} = \sigma_2 w_{2t+1}$$

where y_t is income, z_t is the *persistent* or "permanent income", u_t is transitory changes in income.

- Which one is a martingale (i.e., random walk here)?
- Define the vector of shocks $w_{t+1} = \begin{bmatrix} w_{1t+1} \\ w_{2t+1} \end{bmatrix} \sim N(0_2, I_{2\times 2})$, i.e., iid normal distributed, mean 0, variance 1.

Setup in linear state space form:

Since $x_t = \begin{bmatrix} z_t \\ u_t \end{bmatrix}$, we have:

$$\underbrace{\begin{bmatrix} z_{t+1} \\ u_{t+1} \end{bmatrix}}_{x_{t+1}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} z_t \\ u_t \end{bmatrix}}_{x_t} + \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} w_{1t+1} \\ w_{2t+1} \end{bmatrix}}_{w_{t+1}}$$

$$y_t = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{G} \cdot \underbrace{\begin{bmatrix} z_t \\ u_t \end{bmatrix}}_{x_t}$$

Computing the key matrices:

$$I - \beta A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1-\beta & 0 \\ 0 & 1 \end{bmatrix}$$

$$(I - \beta A)^{-1} = \begin{bmatrix} \frac{1}{1-\beta} & 0 \\ 0 & 1 \end{bmatrix}$$

(Since diagonal matrix, its inverse is just 1 over each element)

$$G(I - \beta A)^{-1} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{1-\beta} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{1-\beta} & 1 \end{bmatrix}$$

Consumption:

Recall:

$$
\begin{aligned}
c_t &= (1-\beta)\left[F_t + \mathbb{E}_t\left(\sum_{j=0}^{\infty}\beta^j y_{t+j}\right)\right] \\
&= (1-\beta)\left[F_t + G(I-\beta A)^{-1}x_t\right] \\
&= (1-\beta)\left[F_t + \begin{bmatrix}\frac{1}{1-\beta} & 1\end{bmatrix}\cdot\begin{bmatrix}z_t \\ u_t\end{bmatrix}\right] \\
&= (1-\beta)\left[F_t + \frac{1}{1-\beta}z_t + u_t\right]
\end{aligned}
$$

$$
\boxed{c_t = (1-\beta)F_t + z_t + (1-\beta)u_t}
$$

Note: The coefficient on u_t is $(1-\beta)$, the marginal propensity to consume (MPC) out of transitory income; the coefficient on z_t is 1, which is the MPC out of permanent income. The marginal propensity to consume out of financial wealth F_t is the same as before.

Consumption changes:

Recall:

$$
\begin{aligned}
c_{t+1} - c_t &= (1-\beta)G(I-\beta A)^{-1}\cdot C\cdot w_{t+1} \\
&= (1-\beta)\begin{bmatrix}\frac{1}{1-\beta} & 1\end{bmatrix}\begin{bmatrix}\sigma_1 & 0 \\ 0 & \sigma_2\end{bmatrix}\cdot\begin{bmatrix}w_{1t+1} \\ w_{2t+1}\end{bmatrix}
\end{aligned}
$$

$$
\boxed{c_{t+1} - c_t = \sigma_1 w_{1t+1} + (1-\beta)\sigma_2 w_{2t+1}}
$$

That is, the consumer consumes all of the permanent shock, and the MPC out of the transitory shock.

Savings:

Recall:

$$
\begin{aligned}
F_{t+1} - F_t &= \frac{1}{\beta}\left[y_t - (1-\beta)\mathbb{E}_t\sum_{j=0}^{\infty}\beta^j y_{t+j}\right] \\
&= \frac{1}{\beta}\left[G\cdot x_t - (1-\beta)G(I-\beta A)^{-1}x_t\right] \\
&= \frac{1}{\beta}G\left[I - (1-\beta)(I-\beta A)^{-1}\right]x_t
\end{aligned}
$$

Computing:

$$G \cdot I = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$(1-\beta)G(I-\beta A)^{-1} = \begin{bmatrix} 1 & 1-\beta \end{bmatrix}$$

Therefore:

$$\begin{aligned} F_{t+1} - F_t &= \frac{1}{\beta}\left[\begin{bmatrix} 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1-\beta \end{bmatrix}\right] \begin{bmatrix} z_t \\ u_t \end{bmatrix} \\ &= \frac{1}{\beta}\begin{bmatrix} 0 & \beta \end{bmatrix} \begin{bmatrix} z_t \\ u_t \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} z_t \\ u_t \end{bmatrix} \end{aligned}$$

$$\boxed{F_{t+1} - F_t = u_t}$$

That is, the consumer spends all of z_t and saves nothing, but saves a fraction of transitory income (which returns on savings to F_{t+1}). The fraction of u_t consumed is the annuity value $\frac{R-1}{R}u_t$ since $R(1 - \frac{R-1}{R})u_t = u_t$ for the rest of the income.

References