Stochastic Asset Pricing and Expected Present Discounted Values

Honours Intermediate Macro

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Asset Pricing with Markov Chains

Stochastic Asset Pricing with Discrete States

Setup:

- Assume a discrete number, $1, \ldots N$, of possible states of the world
- Let P be the transition matrix of the Markov chain for these states, and let $A \equiv P^\top$ (the transpose)
- If π_t is the pmf as a row vector, let $x_t \equiv \pi_t^\top$ be the distribution (pmf) of possible states for the random variable as a column vector

Take the standard forecast, $\pi_{t+1} = \pi_t P$ and take the transpose of both sides to get $x_{t+1} \equiv \pi_{t+1}^\top = (P^\top \pi_t^\top) = Ax_t$. Then we see the forecast j into the future is

$$x_{t+j} = A^j \cdot x_t$$

Let the payoff in each state be $G = \begin{bmatrix} y_1 & \cdots & y_N \end{bmatrix}$, so

$$y_t = \begin{bmatrix} y_1 & \cdots & y_N \end{bmatrix} \cdot \begin{bmatrix} x_{1t} \\ x_{2t} \\ \vdots \\ x_{nt} \end{bmatrix} = G \cdot x_t$$

Given the possible payout states, the random variable Y_t is all of the possible payouts in G with probability x_t. So

$$y_t \equiv \mathbb{E}_t [Y_t] = Gx_t$$

Compare to the **linear state space model**: $x_{t+1} = Ax_t$ and $y_t = Gx_t$.

Example:

- $y_t = y_1$ if $x_{1t} = 1$
- $y_t = y_N$ if $x_{Nt} = 1$
- If 50% chance in each of the first 2 states:

$$y_t = G \cdot x_t = \begin{bmatrix} y_1 & \cdots & y_N \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \frac{1}{2} y_1 + \frac{1}{2} y_2$$

This gives the expected dividends.

Using Markov chains for forecasting:

- $x_{t+j} = A^j x_t$
- $y_t = G \cdot x_t$
- Using the forecast and weighting by the pmf: $y_{t+j} = \mathbb{E}_t \left[Y_{t+j} \right] = Gx_{t+j}$

Asset pricing formula:

$$p_t(x_t) = \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j Y_{t+j} \right] = G \left(\sum_{j=0}^{\infty} \beta^j A^j \right) x_t$$

This is close to our old form (note that we have the transpose $A \equiv P^\top$):

$$\boxed{p(x_t) = G(I - \beta A)^{-1} x_t}$$

Compare to the deterministic formula!

Sequential vs. Recursive Thinking

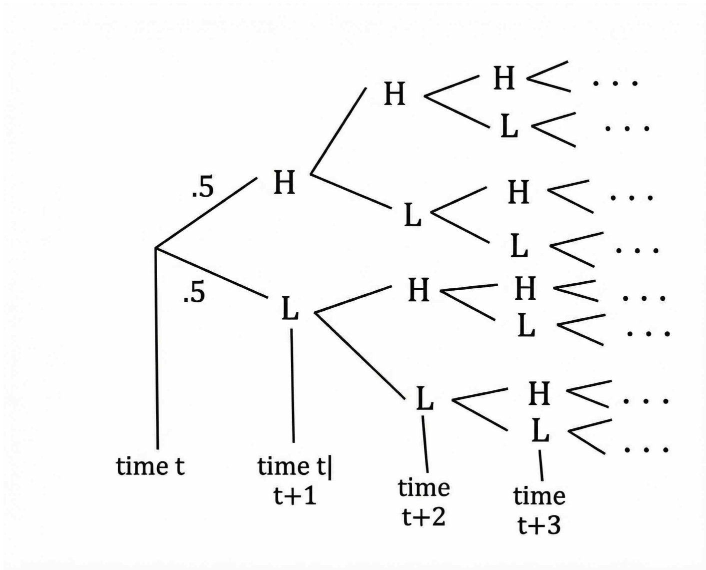

Figure 1: Expected PDV of dividends

An example is an H/L process of dividends:

- Dividends are y_H with probability 0.5 and y_L with probability 0.5 (iid). Can denote these as $\mathbb{P}(H) = \mathbb{P}(L) = 0.5$
- What is the expected present discounted value of payoffs? That is, $p(Y_0) = \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j Y_{t+j} \,\middle|\, Y_0 \right]$?

The figure above shows how complicated this is to think through sequentially. But can we write down a recursive version of this under the assumption that the price should be only a function of the current state?

Define p_H and p_L as the prices in state H vs. L. With this, we can write down a system of two equations and two unknowns.

$$p_H = y_H + \beta \mathbb{E}[p_i \mid H] = y_H + \beta [\mathbb{P}(H) p_H + \mathbb{P}(L) p_L]$$
$$p_L = y_L + \beta \mathbb{E}[p_i \mid L] = y_L + \beta [\mathbb{P}(H) p_H + \mathbb{P}(L) p_L]$$

Stack as vectors:

$$
\begin{aligned}
p &\equiv \begin{bmatrix} p_H & p_L \end{bmatrix} \\
G &\equiv \begin{bmatrix} y_H & y_L \end{bmatrix} \\
A &\equiv \begin{bmatrix} \mathbb{P}(H) & \mathbb{P}(H) \\ \mathbb{P}(L) & \mathbb{P}(L) \end{bmatrix}
\end{aligned}
$$

Then rewrite the system of equations:

$$p = G + \beta p A$$

Rearrange, being careful with the commutative rules of matrices:

$$p(I - \beta A) = G$$

And assuming things are invertible:

$$p = G(I - \beta A)^{-1}$$

In the more general case of a Markov chain, the A becomes the transpose of a Markov chain—as it does in the previous section. Here the columns are identical because the switches between L and H are iid.

To complete the solution, note that this p is a row vector, so if we set x_t as a column vector as above, $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ if H, $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ if L, then we can calculate the price as $p(x_t) = G(I - \beta A)^{-1} x_t$.

Stochastic Asset Pricing with Continuous State Spaces

Information Sets

- Conditional expectation $\mathbb{E}(X|Y)$ means that in forming the expectation of X, can use anything in Y as if known with certainty (i.e., not a random variable)
- $\mathbb{E}_t(C_{t+1})$ is the abbreviation for $\mathbb{E}(C_{t+1} \mid C_t, C_{t-1}, C_{t-2}, \cdots$ and anything else we know at t)
- If first-order Markov, then $\mathbb{E}_t(C_{t+1}) = \mathbb{E}(C_{t+1} \mid C_t)$ (i.e., all info in last state)
- What to choose for the state? Think through necessary information set of an agent.

Properties of Expectations

Key: Expectation is a *linear operator* and can be over scalars, vectors, or matrices.

Some properties of expectations:

- Let a and b be scalar constants, and $\{x_t\}$ and $\{z_t\}$ be scalar random variables
- $\mathbb{E}_t\left[ax_{t+1} + bz_{t+1}\right] = a\mathbb{E}_t\left[x_{t+1}\right] + b\mathbb{E}_t\left[z_{t+1}\right]$
- But, be careful not to apply this for multiplication with other random variables:
  - $\mathbb{E}_t\left[x_{t+1}z_{t+1}\right] \neq \mathbb{E}_t\left[x_{t+1}\right]\mathbb{E}_t\left[z_{t+1}\right]$ in general (true if independent)
  - $\mathbb{E}_t\left[x_{t+1}^2\right] \neq \left(\mathbb{E}_t\left[x_{t+1}\right]\right)^2$ in general. Note x_{t+1} and x_{t+1} are never independent.
  - As always, just be careful to keep the order (i.e., not commutative in general)
  - Of course, if the information is known then the expectation is the value itself: $\mathbb{E}_t\left[x_t\right] = \mathbb{E}\left[x_t|x_t\right] = x_t$
- **Law of iterated expectations:** $\mathbb{E}_t\left[\mathbb{E}_{t+1}\left[x_{t+2}\right]\right] = \mathbb{E}_t\left[x_{t+2}\right]$. Note: time t has less information than that of time $t+1$.

Generalizing, let X_t and Z_t be vector random variables, and A and B be matrices or vectors:

- $\mathbb{E}_t\left[A \cdot X_{t+1} + B \cdot Z_{t+1}\right] = A \cdot \mathbb{E}_t\left[X_{t+1}\right] + B \cdot \mathbb{E}_t\left[Z_{t+1}\right]$
- These also all hold for any conditional expectation as well: $\mathbb{E}_t\left[A \cdot X_{t+1} + B \cdot Z_{t+1} \,|\, Z_t, X_t\right] = A \cdot \mathbb{E}_t\left[X_{t+1} \,|\, Z_t, X_t\right] + B \cdot \mathbb{E}_t\left[Z_{t+1} \,|\, Z_t, X_t\right]$

A Few Tricks with Normal Variables

- If a random variable z is distributed as a normal random variable with mean μ and variance σ^2, it is denoted

$$z \sim N(\mu, \sigma^2)$$

- In terms of expectations, one can show that: $\mathbb{E}\left[z\right] = \mu$ and $\mathbb{E}\left[z^2\right] = \mu^2 + \sigma^2$
- Let $w \sim N(0, 1)$ be a normalized random variable. Then you can show that

$$z = \mu + \sigma w$$

- That is, you can convert any normal random variable to a linear function of a normalized one
- With multivariate normal random variable, $q \in \mathbb{R}^n$, denote its distribution as $q \sim N(\mu, \Sigma)$ where the mean $\mu \in \mathbb{R}^n$ and $\Sigma \in \mathbb{R}^{n \times n}$ is the variance-covariance matrix
- Keeping things simple, if the vector random variable has mean 0 and is independent (i.e., none of the components of the vector have any correlation) then we would write it as $q \sim N(0_n, I_{n \times n})$

Asset Pricing in Our State Space Model

The Deterministic Model

Recall: In the deterministic linear state space, we have

$$\begin{aligned} x_{t+1} &= A \cdot x_t \quad \text{(Evolution)} \\ y_t &= G \cdot x_t \quad \text{(Observation)} \end{aligned}$$

And the asset pricing formula under risk neutrality is:

$$P_t = \sum_{j=0}^{\infty} \beta^j y_{t+j} = G(I - \beta \cdot A)^{-1} \cdot x_t$$

Making This a Stochastic Linear State Space

Add randomness w_{t+1}, an $m \times 1$ vector random variable:

$$\begin{aligned} x_{t+1} &= Ax_t + C \cdot w_{t+1} \quad \text{(Evolution, stochastic)} \\ y_t &= G \cdot x_t \quad \text{(Observation, still noise free)} \end{aligned}$$

where A is $n \times n$ matrix, C is $n \times m$ matrix, w_{t+1} are $m \times 1$ matrices, x is $n \times 1$ vector; G is $1 \times n$ vector, y_t are scalars.

Note: w_{t+1} are independent, identically distributed variables; Gaussian of mean 0, covariance matrix $I_{m \times m}$. Hence, $\mathbb{E}(w_{it+1}) = 0$ for all $i = 1, \cdots m$, and

$$\mathbb{E}(w_{it} w_{i't'}) = \begin{cases} 1 & \text{if } i = i', t = t' \\ 0 & \text{otherwise} \end{cases}$$

Notice that:

$$\mathbb{E}_t(x_{t+1}) = \mathbb{E}_t(A \cdot x_t + Cw_{t+1}) = A \cdot x_t + \underbrace{C \cdot \mathbb{E}_t(w_{t+1})}_{=0} = A \cdot x_t$$

$$\mathbb{E}_t(x_{t+2}) = \mathbb{E}_t \left(A \underbrace{(Ax_t + Cw_{t+1})}_{x_{t+1}} + C \cdot w_{t+2} \right) = \mathbb{E}_t(A^2 x_t + ACw_{t+1} + Cw_{t+2})$$

$$= A^2 x_t + \underbrace{AC\mathbb{E}_t(w_{t+1})}_{=0} + \underbrace{C\mathbb{E}_t(w_{t+2})}_{=0} = A^2 x_t$$

Repeat for $t+3, \cdots$

Forecasting Formulas:

$$\mathbb{E}_t(x_{t+j}) = A^j x_t \quad \text{and} \quad \mathbb{E}_t\left(\sum_{j=0}^{\infty} \beta^j x_{t+j}\right) = (I - \beta \cdot A)^{-1} x_t$$

$$\mathbb{E}_t(y_{t+j}) = G \cdot A^j x_t \quad \text{and} \quad \mathbb{E}_t\left(\sum_{j=0}^{\infty} \beta^j y_{t+j}\right) = G \cdot (I - \beta A)^{-1} x_t$$

Price of a Stochastic Dividend Stream

$$p_t = \mathbb{E}_t\left(\sum_{j=0}^{\infty} \beta^j y_{t+j}\right) + \text{possible bubble} = G(I - \beta A)^{-1} x_t + \text{possible bubble}$$

Or, recursively:

$$p_t = \underbrace{y_t}_{\substack{\text{dividend} \\ \text{today}}} + \beta \cdot \underbrace{\mathbb{E}_t(p_{t+1})}_{\substack{\text{expectation} \\ \text{of price} \\ \text{tomorrow}}}$$

Method (Guess and Verify):

Guess $p_t = H \cdot x_t$, where H is $1 \times n$ vector to be determined, x is $n \times 1$ vector.

Substitute into equation:

$$H \cdot x_t = y_t + \beta \cdot \mathbb{E}_t\left(H x_{t+1}\right)$$

$$H \cdot x_t = G \cdot x_t + \beta H \mathbb{E}_t(A \cdot x_t + C \cdot w_{t+1}) = G \cdot x_t + \beta H A x_t$$

To hold for any x_t:

$$H(I - \beta A) = G \implies H = G(I - \beta A)^{-1}$$

Therefore:

$$\boxed{p_t = G(I - \beta A)^{-1} x_t}$$

Note:

- This is consistent with the EPDV calculation
- Same formula as without random w_{t+1}

Forecast Errors

How far off are the agent's forecasts of $t+1$ given time t information? To do a simple example:

- Let $x_{t+1} = x_t + \sigma w_{t+1}$
- With $w_{t+1} \sim N(0,1)$. That is, $\mathbb{E}_t\left[w_{t+1}\right] = 0$ and $\mathbb{E}_t\left[w_{t+1}^2\right] = 1$.
- This is a trivial linear-Gaussian-state space.

The **expected forecast error** is:

$$\mathbb{E}_t\left[FE_{t+1}\right] \equiv \mathbb{E}_t\left[x_{t+1} - \mathbb{E}_t\left[x_{t+1}\right]\right] = \mathbb{E}_t\left[x_{t+1}\right] - \mathbb{E}_t\left[x_{t+1}\right] = 0$$

No systematic error. What about the variance of the forecast errors?

The variance of a random variable z_t is defined as $\mathbb{V}_t\left(z_{t+1}\right) \equiv \mathbb{E}_t\left[z_{t+1}^2\right] - \left(\mathbb{E}_t\left[z_{t+1}\right]\right)^2$.

So to find the variance of the forecast error:

$$\begin{aligned}
\mathbb{V}_t(FE_{t+1}) &= \mathbb{E}_t\left[FE_{t+1}^2\right] - \left(\mathbb{E}_t\left[FE_{t+1}\right]\right)^2 \\
&= \mathbb{E}_t\left[\left(x_{t+1} - \mathbb{E}_t\left[x_{t+1}\right]\right)^2\right] - 0 \\
&= \mathbb{E}_t\left[\left(x_t + \sigma w_{t+1} - \mathbb{E}_t\left[x_t + \sigma w_{t+1}\right]\right)^2\right] \\
&= \mathbb{E}_t\left[\left(\sigma w_{t+1}\right)^2\right] = \sigma^2
\end{aligned}$$

Linear Gaussian State Space Example

- On average, a worker's productivity, z_t, adds a random draw of $N(\alpha, \sigma^2)$ each period
- Firm productivity q_t adds γ each period, which is deterministic
- Wages are a linear combination: $W_t = \theta z_t + (1-\theta) q_t$

Setup in Linear Gaussian form:

Guess state: $x_t = \begin{bmatrix} z_t \\ q_t \\ 1 \end{bmatrix}$

Note: if $w_{t+1} \sim N(0,1)$, then

$$\alpha + \sigma w_{t+1} \sim N(\alpha, \sigma^2)$$

The state space model is then:

$$\underbrace{\begin{bmatrix} z_{t+1} \\ q_{t+1} \\ 1 \end{bmatrix}}_{x_{t+1}} = \underbrace{\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} z_t \\ q_t \\ 1 \end{bmatrix}}_{x_t} + \underbrace{\begin{bmatrix} \sigma \\ 0 \\ 0 \end{bmatrix} w_{t+1}}_{C \cdot w_{t+1}}$$

$$W_t = \begin{bmatrix} \theta & 1-\theta & 0 \end{bmatrix} \begin{bmatrix} z_t \\ q_t \\ 1 \end{bmatrix} = G \cdot x_t$$

What is the expected PDV of human capital? (i.e., stochastic version of the permanent income calculations)

$$\mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j y_{t+j} \right] = G(I - \beta A)^{-1} x_t$$

Appendices

Stochastic Bubbles

To isolate the bubble term, consider the special case where $y_t = 0$ for all t.

We want to solve $p_t = \beta \mathbb{E}_t(p_{t+1})$, where $\beta = \frac{1}{1+r}$.

Guess: $p_t = C_t \beta^{-t}$, where C_t is a random variable, and $\{C_t\}$ is a **martingale**, that is, satisfies $\mathbb{E}_t(C_{t+1}) = C_t$ (i.e., best forecast of future value is today's value, e.g., random walk).

To verify $p_t = \beta \mathbb{E}_t(p_{t+1})$, substitute our guess:

$$C_t \beta^{-t} = \beta \cdot \mathbb{E}_t(\beta^{-(t+1)} C_{t+1}) = \beta^{-t} \cdot \mathbb{E}_t(C_{t+1}) = \beta^{-t} C_t$$

Verified that $p_t = C_t \beta^{-t}$ satisfies the equation.

Example:

$$C_{t+1} = \begin{cases} \lambda^{-1} C_t & \text{with probability } \lambda \in (0,1) \\ 0 & \text{with probability } 1 - \lambda \end{cases}$$

- Note: $\mathbb{E}_t(C_{t+1}) = \lambda \cdot (\lambda^{-1} C_t) + 0 = C_t$, so this is a martingale
- Note that if at some point $C_{t+j} = 0$, then $C_{t+j+1} = 0$, etc. (i.e., the bubble has popped)

- From any C_0:

$$C_t = \begin{cases} \lambda^{-t} C_0 & \text{if bubble has not popped} \\ 0 & \text{if the bubble has popped} \end{cases}$$

$$p_t = \begin{cases} \beta^{-t} \cdot \lambda^{-t} \cdot C_0 = (\beta\lambda)^{-t} C_0 & \text{until popped} \\ 0 & \text{after the bubble has popped} \end{cases}$$

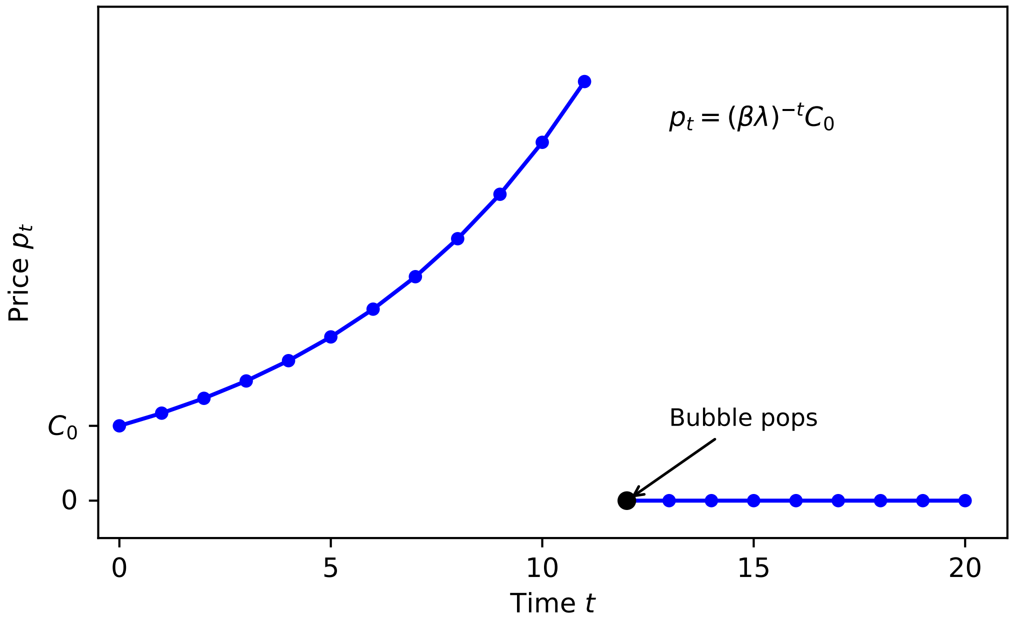

Figure 2: Parameters: $\beta = 0.95$, $\lambda = 0.9$, $C_0 = 1$, with bubble popping at $t = 12$