Markov Chains and Unemployment

Honours Intermediate Macro

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Markov Chains

A model of a stochastic process with a discrete number of states.

Random Variables and Mathematical Expectation

Notation for discrete states:

- $n = 1, \ldots, N$ represents possible "states of the world" (e.g., individual unemployed, employed, …)
- $\pi_n = \mathbb{P}\,(\text{state of the world is } n)$
  - $\pi_n \geq 0$, $\sum_{n=1}^{N} \pi_n = 1$, i.e., the world must be in one of the states
  - Stack as a vector: $\pi \equiv \begin{bmatrix}\pi_1 & \ldots & \pi_N\end{bmatrix}$
- Random variable $Y \in \{y_1, \ldots, y_N\}$
- Values mapping states of the world for r.v. Y: $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$

  e.g., if event n is unemployed, then income if unemployed is y_n

Mathematical Expectation:

- $\mathbb{E}\,[Y] = \sum_{n=1}^{N} \mathbb{P}\,(Y = y_n) y_n = \sum_{n=1}^{N} \pi_n y_n = \pi \cdot y$ (i.e., inner product)
  - Weight the realizations with the probabilities
- e.g., if probability of unemployment is $\pi_1 = 0.1$, income from unemployment insurance is $y_1 = 15{,}000$; probability of employment is $\pi_2 = 0.9$, income from employment is $y_2 = 40{,}000$. Then expected income (or average across states of world):

- $\mathbb{E}[Y] = (0.1 \times 15{,}000) + (0.9 \times 40{,}000)$

- We could use this to find an individual's expected income at some point in the future. Alternatively, we can use this to find averages for a continuum of population—a step towards aggregation.
- e.g., if 10% of population is unemployed at \$15,000 and 90% of population is employed at \$40,000, then the average income is $\mathbb{E}[Y]$

Transitions

Let ϕ = probability to become employed, and α = probability to lose job.

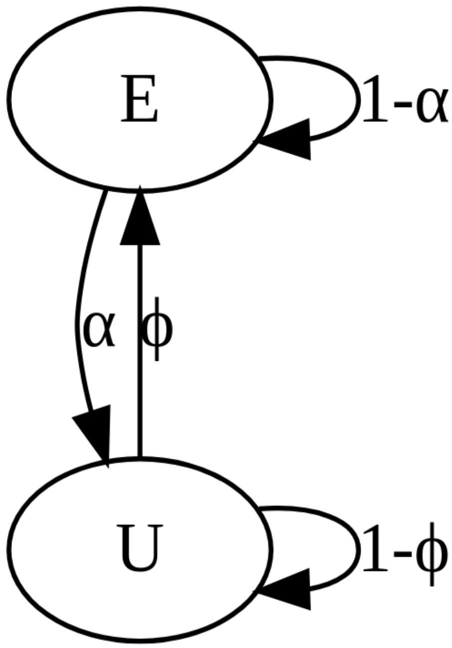

Figure 1: Two-state Markov chain for employment transitions

Let State 1 $\leftrightarrow E$ (Employed), State 2 $\leftrightarrow U$ (Unemployed).

Transition Matrix:

$$P = \begin{array}{c} \\ \end{array} \begin{matrix} & E_{t+1} & U_{t+1} & \\ & \left[\begin{matrix} 1-\alpha \\ \phi \end{matrix}\right. & \left.\begin{matrix} \alpha \\ 1-\phi \end{matrix}\right] & \begin{matrix} E_t \\ U_t \end{matrix} \end{matrix}$$

Let π_t be the probability mass function (pmf) of a random variable of an agent's employment status at time t. This is a probability mass function (pmf) since the possible events are discrete.

- If employed at time 0: $\pi_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$
- If 50% chance of employment at time 3: $\pi_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Evolution of Probability Distribution

Find the evolution of the probability mass function for the random variable with the transition matrix P. A property of Markov chains:

$$\pi_{t+1} = \pi_t \cdot P$$

Careful with the order of the matrix product!

Iterate forward:

$$\pi_{t+j} = \pi_t \cdot P^j$$

Example: Started employed at $t = 0$, i.e., $\pi_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Probability of unemployment/employment at $t = 1$:

$$\pi_1 = \pi_0 \cdot P = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1-\alpha & \alpha \\ \phi & 1-\phi \end{bmatrix} = \begin{bmatrix} 1-\alpha & \alpha \end{bmatrix}$$

At time 2:

$$\pi_2 = \pi_1 \cdot P = \begin{bmatrix} 1-\alpha & \alpha \end{bmatrix} \cdot \begin{bmatrix} 1-\alpha & \alpha \\ \phi & 1-\phi \end{bmatrix} = \begin{bmatrix} (1-\alpha)^2 + \alpha\phi & (1-\alpha)\alpha + \alpha(1-\phi) \end{bmatrix}$$

Interpretation:

$$= \begin{bmatrix} \mathbb{P}\left(E \to E, E \to E\right) + \mathbb{P}\left(E \to U, U \to E\right) & \mathbb{P}\left(E \to E, E \to U\right) + \mathbb{P}\left(E \to U, U \to U\right) \end{bmatrix}$$

Iterating forward:

$$\pi_{t+j} = \pi_t \cdot \underbrace{P \cdot P \cdots P}_{j \text{ times}} = \pi_t \cdot P^j$$

Stationarity and asymptotics. One possibility:

$$\pi_\infty = \lim_{j\to\infty} \pi_{t+j} = \lim_{j\to\infty} \pi_t \cdot P^j$$

Another is to find a π_∞ which doesn't change:

$$\pi_\infty = \pi_\infty P$$

Questions:

- Does a unique limit exist? Is it independent of π_t?
- Is there an absorbing state? (e.g., all end up unemployed forever)
- These answers depend on P.
- In some cases, we will refer to π_∞ as the **stationary distribution**.

Example: Stationary Distribution

Non-Degenerate Stationary Distribution

Consider:

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \phi & 1-\phi \end{bmatrix}, \quad 0 < \alpha < 1, \quad 0 < \phi < 1$$

The stationary random variable π_∞ satisfies:

$$\pi_\infty = \pi_\infty \cdot P$$

i.e., it's the r.v. associated with P such that it doesn't change between periods.

Remark: In linear algebra, the *left* eigenvector associated with the unit eigenvalue.

To find π_∞:

- Use software to find the left eigenvector, or
- Solve system for simple examples:

Let $\bar{\pi}$ = prob of being employed; $\pi_\infty = \begin{bmatrix} \bar{\pi} & 1-\bar{\pi} \end{bmatrix}$.

Then:

$$\begin{bmatrix} \bar{\pi} & 1-\bar{\pi} \end{bmatrix} = \begin{bmatrix} \bar{\pi} & 1-\bar{\pi} \end{bmatrix} \cdot \begin{bmatrix} 1-\alpha & \alpha \\ \phi & 1-\phi \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \bar{\pi} \\ 1-\bar{\pi} \end{bmatrix}' = \begin{bmatrix} \bar{\pi}(1-\alpha) + (1-\bar{\pi})\phi \\ \bar{\pi} \cdot \alpha + (1-\bar{\pi})(1-\phi) \end{bmatrix}'$$

First equation:

$$\bar{\pi} = (1-\alpha)\bar{\pi} - \phi\bar{\pi} + \phi \implies (1-(1-\alpha)+\phi)\bar{\pi} = \phi$$

$$\boxed{\bar{\pi} = \frac{\phi}{\alpha+\phi}} \tag{1}$$

$$\boxed{\pi_\infty = \begin{bmatrix} \frac{\phi}{\alpha+\phi} & \frac{\alpha}{\alpha+\phi} \end{bmatrix}} \tag{2}$$

Second equation: would find identical solution (luckily, since there is only 1 variable and 2 equations).

Unemployment Application

Assume:

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \phi & 1-\phi \end{bmatrix} \quad \begin{matrix} \leftrightarrow E \\ \leftrightarrow U \end{matrix}$$

Invariant Distribution (i.e., "long run"):

$$\bar{\pi} = \mathbb{P}(E), \quad 1-\bar{\pi} = \mathbb{P}(U)$$

solves:

$$\begin{bmatrix} \bar{\pi} & 1-\bar{\pi} \end{bmatrix} \begin{bmatrix} 1-\alpha & \alpha \\ \phi & 1-\phi \end{bmatrix} = \begin{bmatrix} \bar{\pi} & 1-\bar{\pi} \end{bmatrix}$$

From the equation $\bar{\pi}(1-\alpha) + \phi(1-\bar{\pi}) = \bar{\pi}$:

$$\bar{\pi} = \frac{\phi}{\alpha + \phi}, \quad 1 - \bar{\pi} = \frac{\alpha}{\alpha + \phi}$$

Dividing top and bottom by $\phi\alpha$:

$$1 - \bar{\pi} = \frac{1/\phi}{1/\phi + 1/\alpha}$$

Average Unemployment Spell

In each period an unemployed person gets a job with probability ϕ. Otherwise, stays unemployed.

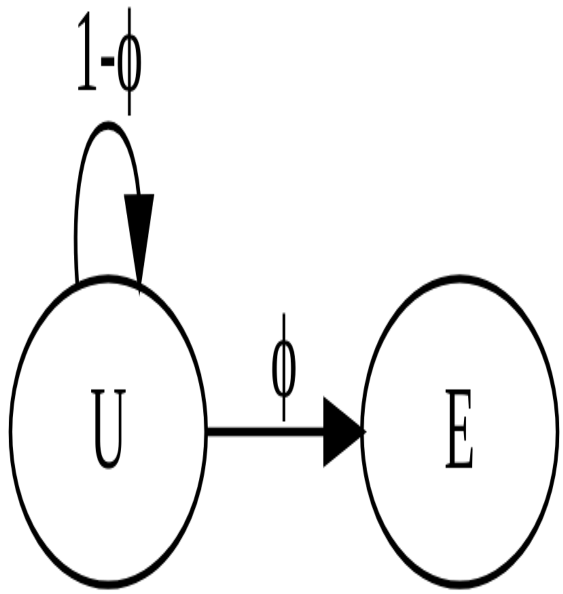

Figure 2: Waiting time to find employment

Let N be the random variable "length of time it takes to find a job". $N = 1$ means 1 period.

Let $p_j = \mathbb{P}(N = j)$.

Then:

- $p_1 = \phi$ (success on first try)
- $p_2 = \phi(1 - \phi)$ (fail, success)
- $p_3 = \phi(1 - \phi)^2$ (fail, fail, success)

$$\boxed{p_j = \phi(1-\phi)^{j-1}}$$

This is a proper probability distribution:

$$\sum_{j=1}^{\infty} p_j = \phi \sum_{j=1}^{\infty} (1-\phi)^{j-1} = \phi \sum_{j=0}^{\infty} (1-\phi)^j = \frac{\phi}{1-(1-\phi)} = 1$$

Another geometric series result:

$$\boxed{\sum_{j=1}^{\infty} j a^{j-1} = \frac{1}{(1-a)^2} \text{ for } |a| < 1}$$

Expected waiting time:

$$\mathbb{E}[N] = \text{expected/mean time to find a job} = \sum_{j=1}^{\infty} j \cdot p_j = \phi \sum_{j=1}^{\infty} j(1-\phi)^{j-1} = \phi \cdot \frac{1}{(1-(1-\phi))^2} = \frac{1}{\phi}$$

So the average number of periods in unemployment $= \frac{1}{\phi}$.

More generally, this is the **mean waiting time for a geometric distribution**: if arrivals happen with probability a, then the expected wait time $= \frac{1}{a}$.

Summary of Formulas

- $\bar{\pi}$ = proportion employed, $1-\bar{\pi}$ = proportion unemployed
- $\bar{\pi} = \dfrac{\phi}{\alpha+\phi}, \quad 1-\bar{\pi} = \dfrac{\alpha}{\alpha+\phi} = \dfrac{1/\phi}{1/\phi + 1/\alpha}$
- $\mathbb{E}[\#\text{ of periods to become employed} \mid \text{start unemployed}] = \frac{1}{\phi}$
- $\mathbb{E}[\#\text{ of periods to become unemployed} \mid \text{start employed}] = \frac{1}{\alpha}$

Calibration Example

Lets say you had the following data;

- Average unemployment duration = 16.8 weeks = 3.87 months
- Civilian unemployment: 4.7%
- Employment / population: 63%
- Labor force / population: 66%
- Civilian population: 231 million
- Civilian labor force: 153 million = 231 × 66% (not institutional military, etc.)
- Unemployment: 7 million = 153 million × 4.7%

Stationary Distribution:

- $1 - \bar{\pi} = 0.047$ (proportion unemployed)
- $\frac{1}{\phi} = 3.87$ (average unemployment length, in months)

From the equation for the stationary distribution:

$$1 - \bar{\pi} = \frac{1/\phi}{1/\phi + 1/\alpha}$$

$$\Rightarrow 0.047 = \frac{3.87}{3.87 + 1/\alpha}$$

Solving for $\frac{1}{\alpha}$:

$$\frac{1}{\alpha} = 78.8$$

i.e., average job length is 78 months.

So, the transition matrix is:

$$P = \begin{bmatrix} 1 - \frac{1}{78.8} & \frac{1}{78.8} \\ \frac{1}{3.87} & 1 - \frac{1}{3.87} \end{bmatrix} \approx \begin{bmatrix} 0.987 & 0.013 \\ 0.258 & 0.742 \end{bmatrix}$$

Stationary distribution:

$$\pi_\infty = \begin{bmatrix} 0.953 & 0.047 \end{bmatrix}$$

Questions:

1. Total jobs destroyed/month: 0.013×146 million ≈ 1.85 million

2. If employed worker today, what is the probability to be employed in j months?

$$\mathbb{P}\left(E \text{ at } j\right) = \begin{bmatrix}1 & 0\end{bmatrix} \cdot \left(\begin{bmatrix}1 & 0\end{bmatrix} P^j\right)'$$

What about as $j \to \infty$? $\mathbb{P}\left(E \text{ at } j \to \infty\right) = \bar{\pi}$

3. The economy is away from its stationary equilibrium: $\pi_0 \neq \pi_\infty$. What is the predicted sequence of unemployment rates?

$$\pi_j = \begin{bmatrix}0 & 1\end{bmatrix} \cdot \left[\pi_0 P^j\right]'$$

Appendices

Absorbing State of Unemployment Back

Let $P = \begin{bmatrix}\alpha & 1-\alpha \\ 0 & 1\end{bmatrix}$,

i.e., α chance to stay employed, and in unemployment never get a job ("absorbing").

Let $\pi_0 = \begin{bmatrix}a & 1-a\end{bmatrix}$.

$$\pi_1 = \pi_0 P = \begin{bmatrix}a & 1-a\end{bmatrix} \begin{bmatrix}\alpha & 1-\alpha \\ 0 & 1\end{bmatrix} = \begin{bmatrix}\alpha a & (1-\alpha)a + 1 - a\end{bmatrix} = \begin{bmatrix}\text{kept job} & \text{lost job or never had one}\end{bmatrix}$$

$$\pi_2 = \pi_1 P = \begin{bmatrix}\alpha a & (1-\alpha)a + 1 - a\end{bmatrix} \begin{bmatrix}\alpha & 1-\alpha \\ 0 & 1\end{bmatrix} = \begin{bmatrix}\alpha^2 a & (1-\alpha)\alpha a + (1-\alpha)a + (1-a)\end{bmatrix}$$

Note: $\alpha^2 a$ represents keeping job twice.

General pattern:

$$\pi_j = \pi_0 \cdot P^j = \begin{bmatrix}\alpha^j \cdot a & 1 - \alpha^j \cdot a\end{bmatrix}$$

$$\lim_{j \to \infty} \pi_j = \begin{bmatrix}0 & 1\end{bmatrix}$$

i.e., all end up unemployed, independent of π_0.

Or via powers of P:

$$P^2 = \begin{bmatrix} \alpha & 1-\alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 1-\alpha \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^2 & \alpha(1-\alpha)+(1-\alpha) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^2 & 1-\alpha^2 \\ 0 & 1 \end{bmatrix}$$

Generalize:

$$P^j = \begin{bmatrix} \alpha^j & 1-\alpha^j \\ 0 & 1 \end{bmatrix}, \quad \lim_{j\to\infty} P^j = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\pi_\infty = \lim_{j\to\infty} \pi_0 P^j = \begin{bmatrix} a & 1-a \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Alternatively, solve $\pi_\infty = \pi_\infty \cdot P$:

$$\begin{bmatrix} \bar{\pi} & 1-\bar{\pi} \end{bmatrix} = \begin{bmatrix} \bar{\pi} & 1-\bar{\pi} \end{bmatrix} \begin{bmatrix} \alpha & 1-\alpha \\ 0 & 1 \end{bmatrix}$$

Equation: $\bar{\pi} = \alpha \cdot \bar{\pi} + 0$

If $\alpha < 1$, then $\bar{\pi} = 0$, so $\pi_\infty = \begin{bmatrix} 0 & 1 \end{bmatrix}$.

No Ergodic Distribution Back

Consider:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

i.e., switch from whatever you had.

$$P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P^j = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } j \text{ even} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \text{if } j \text{ odd} \end{cases}$$

$\lim_{j\to\infty} P^j$ doesn't exist in general.

Alternatively, solve $\pi_\infty = \pi_\infty \cdot P$:

$$\begin{bmatrix} a & 1-a \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & 1-a \end{bmatrix}$$

$$\begin{bmatrix} 1-a & a \end{bmatrix} = \begin{bmatrix} a & 1-a \end{bmatrix} \implies \pi_\infty = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

i.e., must start out with 50/50% probability for the distribution to be stationary.