Linear Difference Equations and Asset Pricing

Honours Intermediate Macro

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Solutions (and Uniqueness) of Difference Equations

From the previous lecture notes, pricing a sequence $\{y_{t+j}\}$ of payoffs:

$$P_t = \sum_{j=0}^{\infty} \beta^j y_{t+j} \quad \text{at time } t \text{ (Sequential formulation)}$$

Can be written:

$$P_t = y_t + \beta P_{t+1}$$

Solving with Guess and Verify

How can we solve a difference equation?

Example: $y_t = \bar{y}$

$$P_t = \bar{y} + \beta P_{t+1} \tag{1}$$

A Guess: $P_t = \bar{P}$, independent of t. Plug in Equation 1:

$$\bar{P} = \bar{y} + \beta \bar{P} \quad \Rightarrow \quad \bar{P} = \frac{\bar{y}}{1-\beta}, \text{ consistent with } P = \sum_{t=0}^{\infty} \beta^t y_t$$

Role of $|\beta| < 1$:

- Keep from "exploding": stability
- Will have equivalent condition for more complicated difference equations

Rational Bubbles

Let $y_t = \bar{y}$ for all t.

Fundamental value:

$$P_t = \sum_{j=0}^{\infty} \beta^j \bar{y} = \frac{\bar{y}}{1-\beta} \quad \text{(unique)}$$

Remember that this solves the recursive problem as well:

$$\frac{\bar{y}}{1-\beta} = \bar{y} + \beta\left(\frac{\bar{y}}{1-\beta}\right) \Rightarrow \text{true!}$$

Is $P_t = \dfrac{\bar{y}}{1-\beta}$ the unique solution to $P_t = \bar{y} + \beta P_{t+1}$? **No!** Like the undetermined coefficient in differential equations.

Example:

$$P_t = \underbrace{\frac{\bar{y}}{1-\beta}}_{\text{fundamental value}} + \underbrace{c\beta^{-t}}_{\text{bubble term}} \quad \text{for any } c$$

Check: $P_t = \bar{y} + \beta P_{t+1}$

$$\frac{\bar{y}}{1-\beta} + c\beta^{-t} = \bar{y} + \beta\left[\frac{\bar{y}}{1-\beta} + c\beta^{-(t+1)}\right] = \bar{y} + \left(\frac{\beta}{1-\beta}\right)\bar{y} + c\beta^{-t} = \frac{\bar{y}}{1-\beta} + c\beta^{-t}$$

So it fulfills the difference equation for any c, t, etc. *Rational* as every agent in the economy would agree on the price, no one needs to be tricked or making a pricing mistake, and there is no arbitrage. An example of a self-fulfilling equilibrium.

Size of the "Rational Bubble"

$$\underbrace{P_0 - P_{fund}}_{\text{difference from fundamental}} = \frac{\bar{y}}{1-\beta} - \frac{\bar{y}}{1-\beta} + c\beta^0 = c$$

Expectations:

- Prices rise because they are expected to rise.
- Self-fulfilling. Will depend on coordination of expectations.
- Is Fiat money a bubble?

Extending our Asset Pricing Model

We will generalize our results to include systems of equations, with dynamics.

Recall: Properties

- Dividend stream y_t
- Discount factor β
- Present discounted value = price: $P = \sum_{t=0}^{\infty} \beta^t y_t$, and if $y_t = \bar{y}$, $P = \bar{y}(1-\beta)^{-1}$
- How to model the evolution of y_t?
  - Will use **systems** of linear difference equations in an underlying state x_t
- **Example:** dividends are a linear function of evolving aggregate and idiosyncratic variables

Recall: Recursive Formulation: $P_t = y_t + \beta P_{t+1}$

Applying to Dynamics

- Let x_t be an n-dimensional vector of states.
- Let A, G be matrices.
- Stack first order difference equations, giving another *canonical form*:

$$\begin{aligned} x_{t+1} &= A \cdot x_t \qquad &(A \text{ is } n \times n \text{ matrix, } x_t \text{ is } n \times 1 \text{ vector}) \\ y_t &= G \cdot x_t \qquad &(G \text{ is } 1 \times n \text{ vector, } y_t \text{ is a scalar, i.e. } 1 \times 1) \end{aligned}$$

- "A" gives evolution of the state, given x_0
- "G" gives observation of the state
  - *"Finding the state is an art"*

Example:

- Asset payoff follows difference equation (not first order!):

$$y_{t+1} = \rho_1 y_t + \rho_2 y_{t-1}$$

- What is the value of this asset at time t?

State:

Guess: $x_t \equiv \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}$, a 2×1 vector.

What is the difference equation for x_t?

$$\underbrace{\begin{bmatrix} y_{t+1} \\ y_t \end{bmatrix}}_{x_{t+1}} = \underbrace{\begin{bmatrix} \rho_1 & \rho_2 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}}_{x_t}$$

And observation:

$$y_t = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}}_{x_t}$$

Therefore, the set of difference equations in our *canonical form* are:

$$\begin{aligned} x_{t+1} &= Ax_t \\ y_t &= Gx_t \end{aligned}$$

Price is:

$$P_t = \sum_{j=0}^{\infty} \beta^j y_{t+j} = \sum_{j=0}^{\infty} \beta^j G \cdot x_{t+j}$$

If $x_{t+1} = A \cdot x_t$, then $x_{t+2} = A \cdot (Ax_t) = A^2 x_t$, and $x_{t+j} = A^j x_t$

$$\Rightarrow P_t = \sum_{j=0}^{\infty} \beta^j G \cdot A^j \cdot x_t = G \cdot \left[\sum_{j=0}^{\infty} (\beta A)^j \right] x_t$$

Remember that if λ is scalar: $\sum_{j=0}^{\infty} (\beta\lambda)^j = (1 - \beta\lambda)^{-1} = \frac{1}{1-\beta\lambda}$.

With matrices and inverses, this is similar: $\sum_{j=0}^{\infty} \beta^j A^j = (I - \beta A)^{-1}$,

where the matrices' dimensions are: $A : n \times n$, $I = n \times n$ identity, $(I - \beta A)^{-1} : n \times n$

$$\boxed{P_t = G\,(I - \beta A)^{-1}\, x_t} \quad \text{(very important, memorize!)}$$

- Asset pricing formula for first-order linear difference equations.
- Summary of sizes:

- P_t: 1×1 scalar
- G: $1 \times n$ vector
- A: $n \times n$ matrix
- I: $n \times n$ identity matrix
- β: 1×1 scalar
- x_t: $n \times 1$ state vector

Stability

- Recall in the example with $x_t = \lambda^t$ that $|\beta\lambda| < 1$ for the series to converge.
- For matrix equations, need a similar condition where eigenvalues of βA are all < 1, or $\max|\text{eig}(A)| < \frac{1}{\beta}$
- Can use software to check the eigenvalues.

Appendices

Connection to Differential Equations

Difference equations are just differential equations in discrete time.

- Let $y(t)$ be the **flow** dividends, a function of t.
- Let r be the instantaneous interest rate.
- Let the length of a period be Δ, and take the limit as it **goes to 0**.
- Dividends over Δ period $\approx \Delta y(t) \equiv y_t(\Delta)$
- Discounting over Δ period $\approx 1 - \Delta r \equiv \beta(\Delta)$

The difference equation is: $P_t = y_t + \beta P_{t+1}$.

Using the above: Let function $p(t)$ be the price of asset:

$$p(t) = \Delta \cdot y(t) + (1 - \Delta r) \cdot p(t + \Delta)$$

Rearrange:

$$\Delta r \cdot p(t + \Delta) = \Delta \cdot y(t) + p(t + \Delta) - p(t)$$

$$\Rightarrow rp(t + \Delta) = y(t) + \frac{p(t + \Delta) - p(t)}{\Delta}$$

Take limit as $\Delta \to 0$, i.e. discrete $\to$ continuous t

$$\partial p(t) = \frac{p(t+\Delta) - p(t)}{\Delta} \quad \text{(definition of a derivative)}$$

where $\partial p(t) = \frac{d}{dt}p(t)$

$$\Rightarrow \underbrace{rp(t)}_{\text{opportunity cost of buying a unit of the asset}} = \underbrace{y(t)}_{\text{flow dividends}} + \underbrace{\partial p(t)}_{\text{capital gains}}$$

- Consider this pricing equation and arbitrage:
  - What if $rp(t) < y(t) + \partial p(t)$ instead of being an equation?

Popping Bubbles

In our discrete time model, keep $y_t = \bar{y}$ deterministic for simplicity:

- Let the bubble term have a chance of popping each period.
- Therefore, prices are a random variable.
- Linear asset pricing if random:

$$P_t = y_t + \beta \mathbb{E}_t \left[P_{t+1}\right] \quad \text{(Expected value of } P_{t+1} \text{ given information at } t\text{)}$$

Bubble Evolution

$$\text{Let } C_{t+1} = \begin{cases} \frac{1}{\lambda}C_t & \text{with prob. } \lambda \in (0,1) \\ 0 & \text{with prob. } 1-\lambda \end{cases}$$

i.e., C_t multiplied by $\frac{1}{\lambda}$ each time until bubble breaks. Then $C_t = 0$ for all t.

Note:

$$\mathbb{E}_t \left[C_{t+1}\right] = \lambda \left(\frac{1}{\lambda}C_t\right) + (1-\lambda) \cdot 0 = C_t$$

If $\mathbb{E}_t \left[y_{t+1}\right] = y_t$, then this term is called a *martingale*.

Price Level

We can check that for any C_0:

$$P_t = \begin{cases} \frac{\bar{y}}{1-\beta} + (\beta\lambda)^{-t} \cdot C_0 & \text{if bubble hasn't popped} \\ \frac{\bar{y}}{1-\beta} & \text{after bubble pops} \end{cases}$$

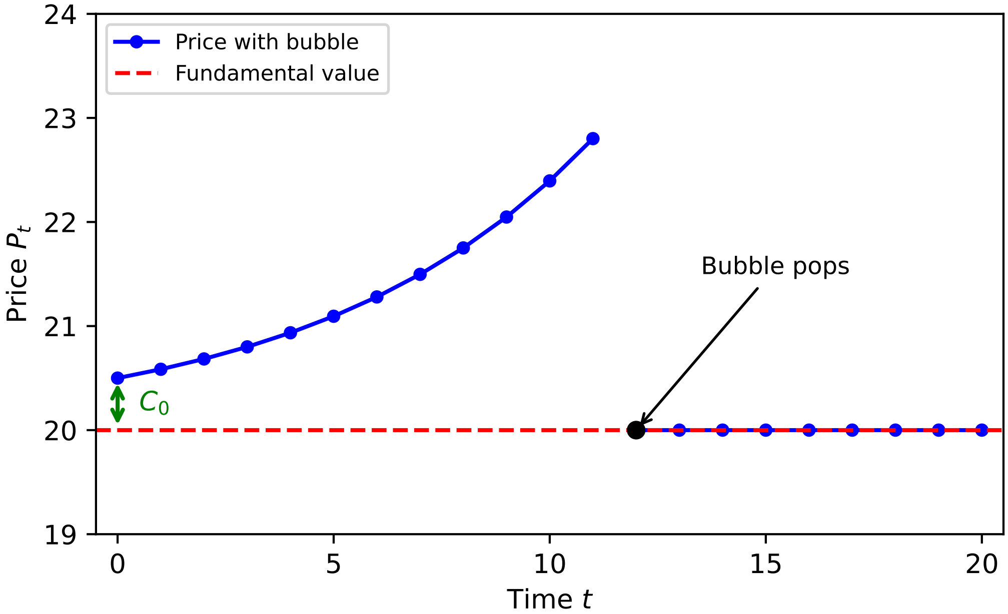

Figure 1: Parameters: $\bar{y} = 1$, $\beta = 0.95$, $\lambda = 0.9$, $C_0 = 0.5$, with $C_{12} = 0$