ECON307 Problem Set 4

Jesse Perla

Question 1

Let $y_t \in \mathbb{R}$ be a sequence of dividends such that

$$y_{t+1} = y_t + w_{t+1}$$

where $w_{t+1} \sim N(0, \sigma^2)$ for some $\sigma > 0$, i.e., $\mathbb{E}_t\left[w_{t+1}\right] = 0$ and $\mathbb{E}_t\left[w_{t+1}^2\right] = \sigma^2$. An agent prices an asset as the expected PDV of dividends,

$$p_t = \mathbb{E}_t\left[\sum_{j=0}^{\infty} \beta^j y_{t+j}\right]$$

Define the forecast error as the deviation in the expected price tomorrow calculated at time t from the actual price tomorrow, i.e.,

$$FE_{t+1|t} \equiv p_{t+1} - \mathbb{E}_t\left[p_{t+1}\right]$$

Question 1.1

Setup in our canonical Linear Gaussian State Space model.

Question 1.2

Solve for p_t in terms of y_t and model intrinsics.

Question 1.3

Find the **expected forecast error**: $\mathbb{E}_t\left[FE_{t+1|t}\right] = \mathbb{E}_t\left[p_{t+1} - \mathbb{E}_t\left[p_{t+1}\right]\right]$

Question 1.4

Find the **variance of forecast errors**:

$$\begin{aligned}\mathbb{V}_t\left(FE_{t+1|t}\right) &\equiv \mathbb{E}_t\left[FE^2_{t+1|t}\right] - \left(\mathbb{E}_t\left[FE_{t+1|t}\right]\right)^2 \\ &= \mathbb{E}_t\left[\left(p_{t+1} - \mathbb{E}_t\left[p_{t+1}\right]\right)^2\right] - \left(\mathbb{E}_t\left[p_{t+1} - \mathbb{E}_t\left[p_{t+1}\right]\right]\right)^2\end{aligned}$$

Question 1.5

Setup the problem recursively as p_t defined in terms of p_{t+1}. Test, using your solution from earlier in the question, if the stochastic process p_t is a martingale (i.e., just a random walk in this case)? Recall that a martingale is a process such that $\mathbb{E}_t\left[p_{t+1}\right] = p_t$. Give any intuition you can on this result.

Question 2

Consider a variation of the example in class where the government makes a surprise announcement that it will borrow money, give the money to the consumers as a "stimulus", and eventually pay it back through taxing consumers (i.e., no chance of a government default).

The agent's income is $y_t = \delta^t$ for $\delta > 1$, and assume that: $\beta R = 1$, $u'(c) > 0$, and $u''(c) < 0$. Furthermore, assume that the consumer and government face the same interest rate $R > 0$. Let $F_0 = 0$.

The government makes its announcement between time 0 and time 1 (i.e., after the consumer has already chosen c_0 and F_1 thinking that their income will follow y_t). The precise announcement is that at time 1, consumers are given $\alpha > 0$ as *extra* income as a stimulus (thereby increasing their y_1 from what they had previously anticipated). That is, income is now

$$y_1 = \delta + \alpha$$

Instead of paying back deterministically, the government will pay back the loan at period k (which will be stochastic). To pay the loan, the government taxes the total value of the loan + interest. For example, if they paid it off in period k, then the labor income of a consumer at period k would be

$$y_k = \delta^k - \alpha R^{k-1}$$

Otherwise, the consumer's income follows the same y_t process. While the consumer doesn't know exactly when the loan will be repaid, they know the correct distribution of payment dates upon the announcement:

$$\mathbb{P}\left(\text{pay at } k\right) = p(k) \geq 0$$

where $\sum_{k=2}^{\infty} p(k) = 1$.

Question 2.1

First, assuming the standard permanent income model, calculate the optimal sequence $\{c_t\}_{t=0}^{\infty}$ at $t = 0$, before the government announces the policy. Note that at this point, the consumer believes they have a deterministic income stream and that the government is not going to borrow or tax.

Question 2.2

After the surprise announcement, what is the new optimal path of consumption chosen? (*Hint: at time* 1 *the consumer's income is now stochastic, but it is very linear and simple.*) What is $c_1 - c_0$? Interpret the effect of the stimulus on consumption.

Question 2.3

Does it matter if the consumer knows the true distribution of payment dates, or the timing of the taxes to pay for the loan?

Question 3

A consumer's optimal decision rule for consumption satisfies

$$c_t = (1-\beta)\Big[F_t + \mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j y_{t+j}\right]\Big] \tag{1}$$

where c_t is consumption, β^{-1} is the gross one-period interest rate (i.e., $\beta R = 1$), which is constant over time, y_t is the consumer's income at time t, F_t is the consumer's financial assets at the beginning of t, and $\mathbb{E}_t\left[\cdot\right]$ means the best forecast of $(\cdot)$, conditional on information that the consumer knows at t. At time t, assume that the consumer knows current and past values of y_t's, but not future values. The consumer's labor income follows the random process

$$y_{t+1} = \delta_0 + \delta_1 y_t + \delta_2 y_{t-1} + \sigma \epsilon_{t+1}$$

where $\{\epsilon_{t+1}\}_{t=0}^{\infty}$ is an independently and identically distributed (iid) sequence of scalar normally distributed random variables, each with mean 0 and variance 1. (Please make whatever assumptions you want about δ_1 and δ_2 in order to make the subsequent questions meaningful.)

Question 3.1

Given available information at time t, give an expression for the consumer's expected income j periods into the future, $\mathbb{E}_t\left[y_{t+j}\right]$.

Question 3.2

Find an expression for the consumer's decision rule of the form

$$c_t = (1-\beta)\Big[F_t + \alpha_0 + \alpha_1 y_t + \alpha_2 y_{t-1}\Big]$$

Please describe how to find formulas for $\alpha_0, \alpha_1, \alpha_2$.

Question 3.3

Measured in constant 2015 dollars, the changes in consumption for this consumer over the last year (which started out better than it ended) were as follows:

Quarter	$c_t - c_{t-1}$
I	1000
II	0
III	0
IV	-4000

What can you infer from these consumption change numbers, if anything, about the consumer's past, present, and future labor income? Can you interpret these in the context of "surprise"?

Formula Sheet

Infinite Sums

$$\sum_{j=0}^{\infty} a^j = \frac{1}{1-a}, \quad \sum_{j=0}^{T} a^j = \frac{1-a^{T+1}}{1-a}$$

$$\sum_{j=1}^{\infty} j\, a^{j-1} = \frac{1}{(1-a)^2}$$

Linear State Space Model (LSS)

$$\begin{aligned} x_{t+1} &= A\, x_t \\ y_t &= G\, x_t \end{aligned}$$

LSS PDV

$$\sum_{j=0}^{\infty} \beta^j y_{t+j} = G\,(I - \beta A)^{-1} x_t$$

Permanent Income Model (PIM)

$$\begin{aligned} &\max_{\{c_t,\, F_{t+1}\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t u(c_t) \\ &\text{s.t.} \quad F_{t+1} = R\Big(F_t + y_t - c_t\Big), \quad t = 0, \dots \end{aligned}$$

PIM Lifetime Budget Constraint

$$\sum_{j=0}^{\infty} \left(\frac{1}{R}\right)^j c_{t+j} = F_t + \sum_{j=0}^{\infty} \left(\frac{1}{R}\right)^j y_{t+j}$$

PIM Euler Equation

$$u'(c_t) = \beta R\, u'(c_{t+1})$$

PIM Solution with $\beta R = 1$

$$\bar{c} = (1-\beta)\Big[F_t + \sum_{j=0}^{\infty} \beta^j y_{t+j}\Big]$$

PIM with $F_{t+1} \geq 0$ Euler Equation

$$\begin{aligned} u'(c_t) &= \beta R\, u'(c_{t+1}) && \text{if } F_{t+1} > 0 \\ u'(c_t) &\geq \beta R\, u'(c_{t+1}) && \text{if } F_{t+1} = 0 \text{ and } c_t = F_t + y_t \end{aligned}$$

Expected Value for Discrete RV

$$\mathbb{E}\left[Y\right] = \sum_{n=1}^{N} \mathbb{P}\Big(Y = y_n\Big) y_n = \sum_{n=1}^{N} \pi_n y_n = \pi \cdot y$$

Evolution of PMF for Markov Chain

$$\pi_{t+j} = \pi_t\, P^j, \quad \text{with } P_{ij} = \mathbb{P}\left[Y_{t+1} = j \mid Y_t = i\right]$$

Asset Pricing with Markov Chain

$$p_t(x_t) = \mathbb{E}_t\Big[\sum_{j=0}^{\infty} \beta^j Y_{t+j}\Big] = G\Big(I - \beta P^\top\Big)^{-1} x_t$$

where $x_t = \pi_t^\top$ and $G = \begin{bmatrix} y_1 & \cdots & y_N \end{bmatrix}$.

Normal Random Variable Linearity

$$\text{If } z \sim N(\mu, \sigma^2), \text{ then } z = \mu + \sigma w, \quad w \sim N(0,1)$$

Linear Gaussian State Space (LGSS)

$$\begin{aligned} x_{t+1} &= A\, x_t + C\, w_{t+1}, \quad w_{t+1} \sim N(0, I) \\ y_t &= G\, x_t \end{aligned}$$

LGSS Forecasting

$$\mathbb{E}_t\left[x_{t+j}\right] = A^j x_t$$
$$\mathbb{E}_t\left[y_{t+j}\right] = G\,A^j x_t$$
$$\mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j y_{t+j}\right] = G\,(I-\beta A)^{-1}x_t$$

Stochastic PIM (SPIM) Euler Equation

$$u'(c_t) = \beta R\,\mathbb{E}_t\Big[u'(c_{t+1})\Big]$$

SPIM Consumption with $\beta R = 1$

$$c_t \cong (1-\beta)\Big(F_t + \mathbb{E}_t\Big[\sum_{j=0}^{\infty}\beta^j y_{t+j}\Big]\Big)$$

SPIM Consumption Change with $\beta R = 1$

$$c_{t+1} - c_t \cong (1-\beta)\sum_{j=0}^{\infty}\beta^j\Big(\mathbb{E}_{t+1}\left[y_{t+j+1}\right] - \mathbb{E}_t\left[y_{t+j+1}\right]\Big)$$

SPIM Consumption Change with $\beta R = 1$ and LGSS

$$c_{t+1} - c_t \cong (1-\beta)\,G\,(I-\beta A)^{-1}C\,w_{t+1}$$