Problem Set 0: Math Review

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Question 1: Matrix and Vector Multiplication

Calculate the following matrices and matrix-vector multiplication. *(No need to hand this question in. This is just for your own practice.)*

Question 1.1

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 5 & 2 \end{bmatrix}$$

Question 1.2

$$\begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 1 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 5 \\ 1 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix}$$

Question 1.3

$$\begin{bmatrix} 5 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 7 & 1 \\ 0 & 4 & 5 \\ 2 & 3 & 4 \end{bmatrix}$$

Question 1.4

$$\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Question 2: Solving Matrix Equations

Being **formal and explicit** in the rules of matrix algebra (e.g. when things are commutative, distributive, when you require invertibility, etc.) solve the following equations for $x \in \mathbb{R}^N$, with vector $b \in \mathbb{R}^N$, matrices A, D, Q, R all $\mathbb{R}^{N \times N}$, and scalar $m \in \mathbb{R}$.

Question 2.1

$$Ax + \left(x^\top D\right)^\top = b$$

Question 2.2

$$Q^{-1}\left(A\,x\,m + mb\right) = Rx$$

Question 3: Linear Systems and Matrix Form

Transform the following linear equations into a linear system with matrices/vectors.

Question 3.1

$$\begin{cases} 2x + 3y = 2 \\ x - 2y = -1 \end{cases} \text{ where } \{x, y\} \text{ are variables}$$

Question 3.2

$$\begin{cases} 2x - 3y = 1 \\ 3x + my = -2 \end{cases} \text{ where } \{x, y\} \text{ are variables}$$

Question 3.3

$$\begin{cases} 2a + b = 1 \\ 3b + 4c = 2 \\ -2a + c = 0 \end{cases} \text{ where } \{a, b, c\} \text{ are variables}$$

Question 3.4

$$\begin{cases} a + b = -3 \\ c - 2 - 4b = 0 \end{cases} \text{ where } \{a, b, c\} \text{ are variables (this will not be of full rank)}$$

Question 4: Linear Transformations

Question 4.1

Find a linear transformation $G \in \mathbb{R}^2$ such that $G \cdot \begin{bmatrix} a & b \end{bmatrix}^\top$ always returns the second element, b.

Question 4.2

Find $H \in \mathbb{R}^{1\times 2}$ such that $H \begin{bmatrix} x & y \end{bmatrix}^\top$ returns the sum $x + y$.

Question 4.3

Find a 2×2 matrix M such that $M \begin{bmatrix} p & q \end{bmatrix}^\top = \begin{bmatrix} q & p \end{bmatrix}^\top$ (i.e., swaps the two elements).

Question 4.4

Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that doubles the first coordinate and leaves the second unchanged. Write a matrix representation of T.

Question 5: Orthogonality

Question 5.1

Find a vector $x \in \mathbb{R}^2$ such that $x \cdot \begin{bmatrix} 1 & 2 \end{bmatrix}^\top = 0$.

Question 5.2

Find a vector $x \in \mathbb{R}^2$ such that $x \cdot \begin{bmatrix} 0 & 1 \end{bmatrix}^\top = 1$.

Question 5.3

Given vectors x^1 and x^2 which may have different norms (i.e., lengths), how can you use the norm and inner product to test if they are orthogonal? Collinear?

Question 6: Undetermined Coefficients

Use undetermined coefficients to solve the following functional and difference equations.

Remember the notation $f'(z) \equiv \frac{df(z)}{dz}$. *You don't need to know anything about differential equations to do this problem.*

Question 6.1

Take a simple linear ODE: $f'(z) = f(z)$. Guess that $f(z) = C_1 e^z + C_2$ and use undetermined coefficients to solve for C_1 and C_2.

Question 6.2

Take the functional equation $[f(z)]^2 = z^2 + 2z + 1$. Guess that the solution is of the form $f(z) = C_1 z + C_2$. Use undetermined coefficients to find C_1 and C_2.

Question 6.3

Take the difference equation $z_{t+1} = g z_t$. Guess $z_t = C_1 C_2^t + C_3$. Show that C_1 is indeterminate and find C_2 and C_3. What if we add subject to $z_0 = A$? Show how this pins down C_1.

Question 7: Probability and Expectations

Let X and Y be random variables such that $X \in \{0, 1\}$ and $Y \in \{1, 2\}$. These are correlated such that

$$\begin{aligned}
\mathbb{P}(X = 0 \text{ and } Y = 1) &= 0.1 \\
\mathbb{P}(X = 0 \text{ and } Y = 2) &= 0.3 \\
\mathbb{P}(X = 1 \text{ and } Y = 1) &= 0.4 \\
\mathbb{P}(X = 1 \text{ and } Y = 2) &= 0.2
\end{aligned}$$

Make sure to show the correct setup with numbers in the equations, but I don't need to see intermediate steps in the calculation after that.

Question 7.1

Calculate the conditional probabilities $\mathbb{P}(X = 0 \,|Y = 1)$ and $\mathbb{P}(X = 1 \,|Y = 1)$.

Question 7.2

Calculate the unconditional expectations $\mathbb{E}[X]$, $\mathbb{E}[Y]$, and $\mathbb{E}[XY]$.

Question 7.3

Calculate the conditional expectations $\mathbb{E}[X \mid Y = 1]$, $\mathbb{E}[X \mid Y = 2]$, and $\mathbb{E}[XY \mid Y = 1]$.

Question 7.4

Calculate $\mathbb{E}[X \mid Y = 1 \text{ or } Y = 2]$.

Question 8: Statistical Independence

Consider a worker who may be employed or unemployed (E or U), and an economy that may be good or bad (G or B). Let X be the random variable of the worker's employment status and Y be the random variable of the aggregate economy. Now assume we know the following probabilities:

- $\mathbb{P}(X = E \text{ and } Y = G) = 0.5 + \gamma$
- $\mathbb{P}(X = U \text{ and } Y = G) = 0.1$
- $\mathbb{P}(X = E \text{ and } Y = B) = 0.3 - \gamma$
- $\mathbb{P}(X = U \text{ and } Y = B) = 0.1$

for some parameter $|\gamma| < 0.3$.

Question 8.1

Find conditions on γ for statistical independence of the individual's unemployment and the economy's state, and interpret.

Question 9: Constrained Optimization

Solve the following optimization problems. Please be **explicit** in your transformation to our canonical form of constrained optimization, and be **formal** with Lagrange multipliers, first order necessary conditions, inequalities, etc.

Question 9.1

$$\max_x \left\{-x^2 + 2x + 3\right\} \quad \text{s.t. } x \geq 0$$

Question 9.2

$$\min_x \{2x + 3\} \quad \text{s.t. } x \leq 1$$