Wealth Distribution, Firm Dynamics, and Inequality

Undergraduate Computational Macro

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Overview

Motivation and Materials

- In this lecture, we will introduce wealth and income distributions and the dynamics that lead to their shape
- In addition, we will investigate the role of multiplicative growth in firm dynamics
- This will also let us explore heavy-tailed distributions and get a sense of when they will influence inequality

Materials

- Adapted from QuantEcon lectures coauthored with John Stachurski and Thomas J. Sargent
  - → Wealth Distribution Dynamics
- Other references in the Python lectures by Stachurski and Sargent
  - → Heavy-Tailed Distributions
  - → Kesten Processes

```
using Distributions, Plots, LaTeXStrings, LinearAlgebra, BenchmarkTools
using Plots.PlotMeasures, StatsPlots
default(;legendfontsize=16, linewidth=2, tickfontsize=12,
         bottom_margin=15mm)
```

Tails of Distributions

Counter-CDFs

- The counter-CDF is the probability that the value is above a certain value
- It is the complement of the CDF

$$\mathbb{P}(X > x) = 1 - \mathbb{P}(X \leq x)$$

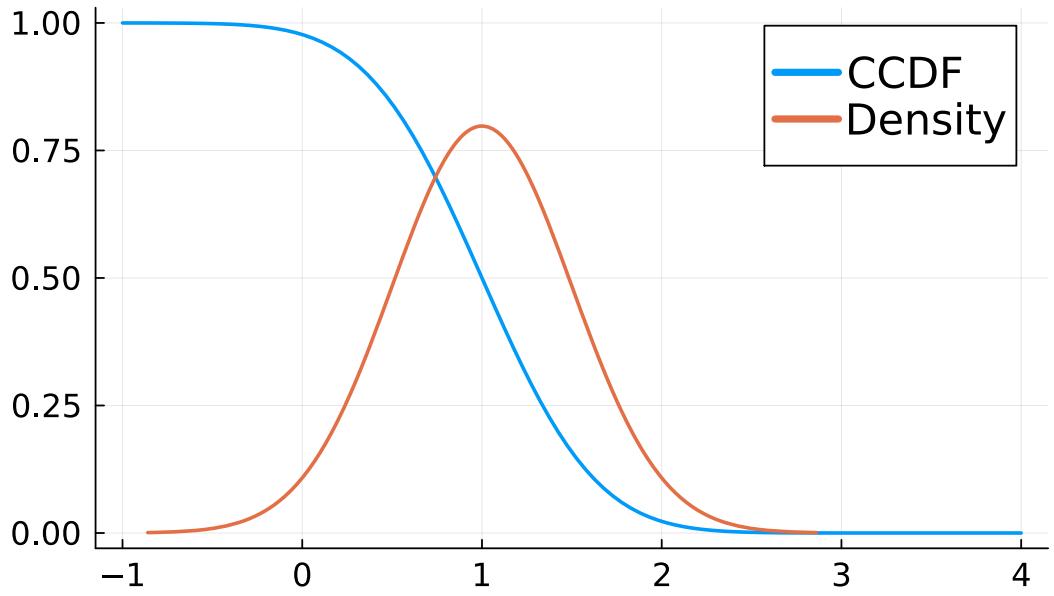
- Or, if there is a density $f(x)$, then

$$\int_x^\infty f(x)dx$$

CCDF for the Normal

- The lognormal distribution is often used to model returns

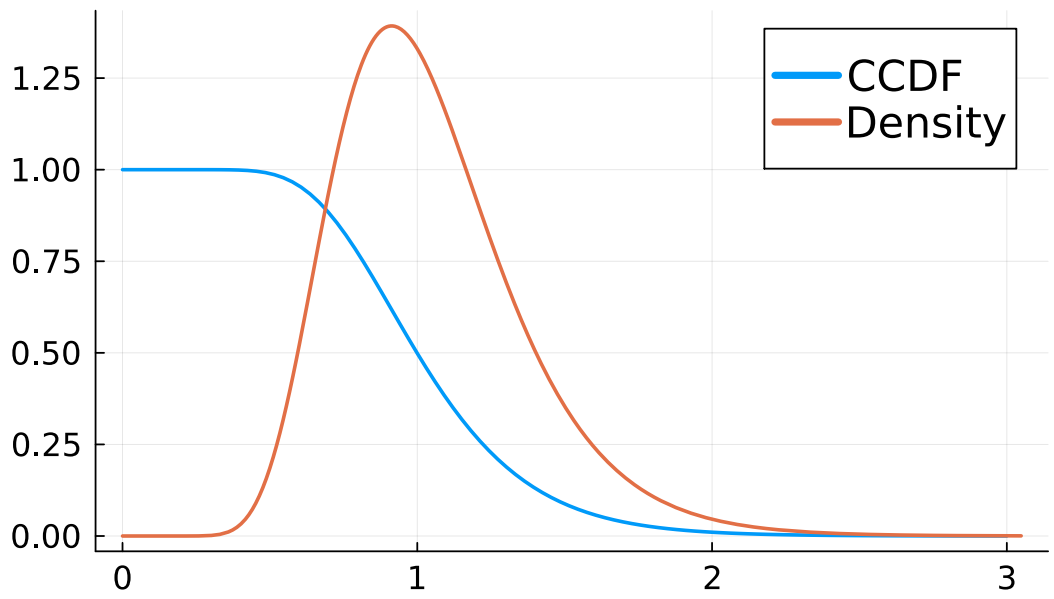
```
mu = 1.0
sigma = 0.5
dist = Normal(mu, sigma)
x = range(-1.0, 4.0, length = 100)
plot(x, 1 .- cdf(dist, x);
     label = "CCDF", size = (600, 400))
plot!(dist; label = "Density")
```



CCDF for the LogNormal

- The lognormal distribution is often used to model returns

```
mu = -0.001
sigma = 0.3
dist = LogNormal(mu, sigma)
x = range(0.0, 3.0, length = 100)
plot(x, 1 .- cdf(dist, x);
     label = "CCDF", size = (600, 400))
plot!(dist; label = "Density")
```



The Pareto Distribution

- Those distributions have relatively few large (or small) values
- The Pareto distribution has a heavy tail
  - → It is often used to model wealth, city sizes, and other phenomena where there are many small values and a few large ones
- The density, given min-value x_m and a shape parameter α

$$f(x) = \frac{\alpha x_m^\alpha}{x^{\alpha+1}}, \text{for all } x \geq x_m$$

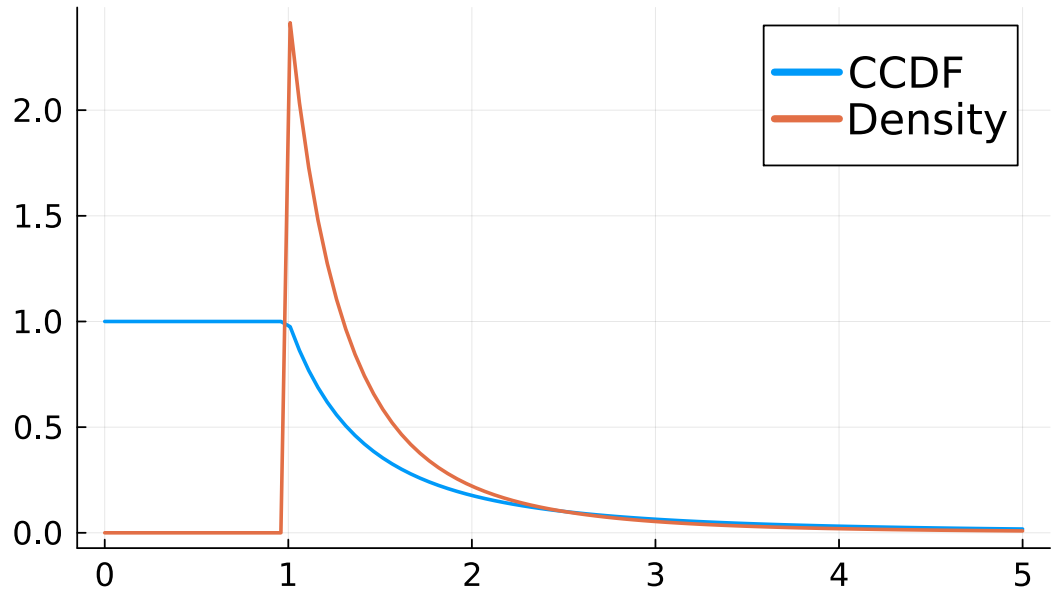
- CDF is $F(x) = 1 - \left(\frac{x}{x_m}\right)^{-\alpha}$, CCDF = $\left(\frac{x}{x_m}\right)^{-\alpha}$, for $x \geq x_m$

CCDF for the Pareto with $\alpha = 2.5$

- As you can see, the CCDF drops fairly slowly

```
1 x_m = 1.0
2 alpha = 2.5
3 x = range(0.0, 5.0, length = 100)
4 dist = Pareto(alpha, x_m)
5 @show mean(dist)
6 @show var(dist)
7 plot(x, 1 .- cdf(dist, x);
8      label = "CCDF", size = (600, 400))
9 plot!(x, dist; label = "Density")
```

```
mean(dist) = 1.6666666666666667
var(dist) = 2.2222222222222223
```

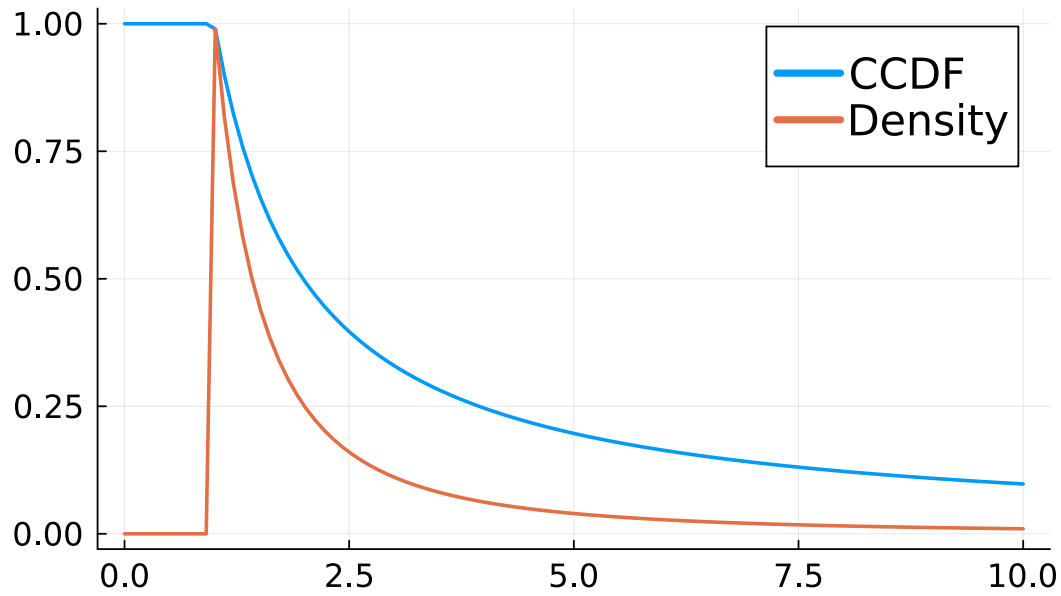


CCDF for the Pareto with $\alpha = 1.0$

- With a smaller α it is even heavier tailed, and doesn't have a variance

```
1 x_m = 1.0
2 alpha = 1.01
3 x = range(0.0, 10.0, length = 100)
4 dist = Pareto(alpha, x_m)
5 @show mean(dist)
6 @show var(dist)
7 plot(x, 1 .- cdf(dist, x);
8      label = "CCDF", size = (600, 400))
9 plot!(x, dist; label = "Density")
```

```
mean(dist) = 100.99999999999991
var(dist) = Inf
```



Log-Log Plots

- The CCDF is often plotted on a log-log scale. i.e. $\log(x)$ vs. $\log(1 - F(x))$
- Taking the log of the probability lets us see the speed that the tail drops off
- For the Pareto distribution, the CCDF is

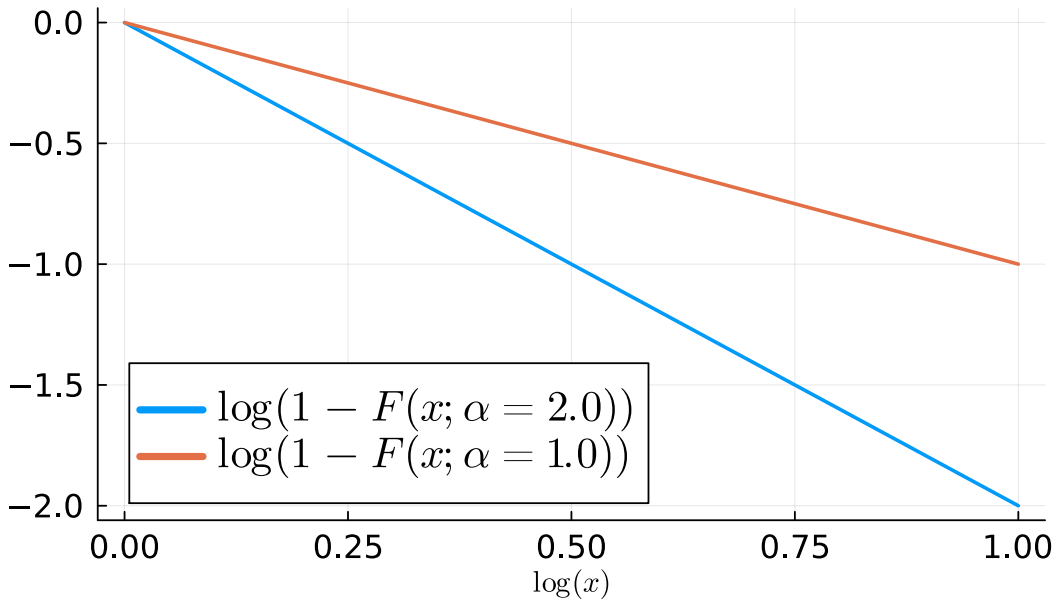
$$\left(\frac{x}{x_m} \right)^{-\alpha}$$

→ Taking the log of this gives

$$\alpha \log(x_m) - \alpha \log(x)$$

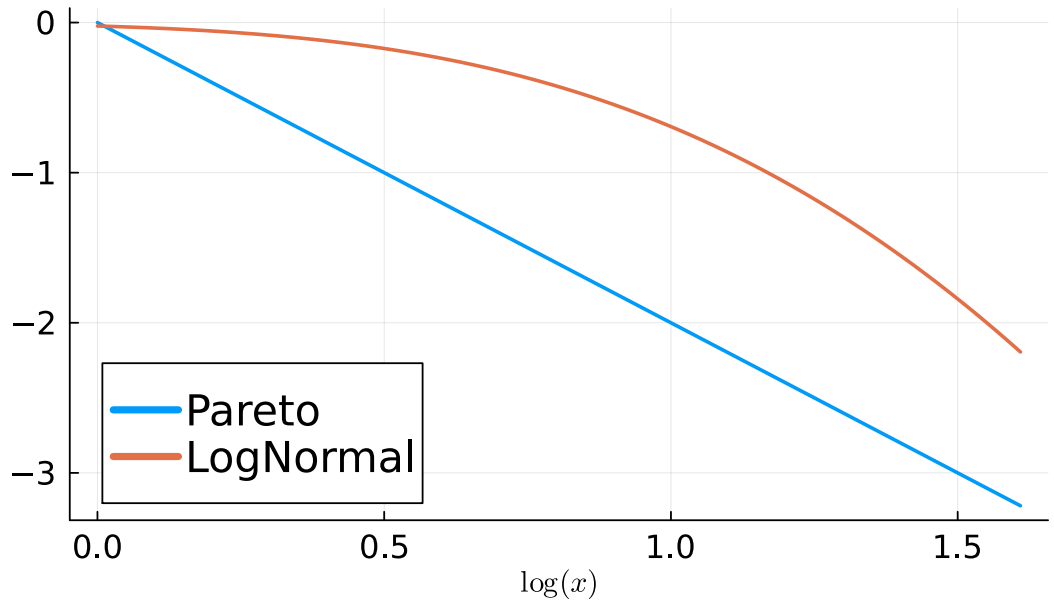
Log-Log Plot for the Pareto

```
x_m = 1.0
x = range(1.0, exp(1), length = 100)
plot(log.(x),
     log.(1 .- cdf(Pareto(2.0, x_m), x));
     label = L"\log(1-F(x;\alpha=2.0))",
     xlabel = L"\log(x)", size = (600, 400),
     legend = :bottomleft)
plot!(log.(x),
     log.(1 .- cdf(Pareto(1.0, x_m), x));
     label = L"\log(1-F(x;\alpha=1.0))")
```



Log-Log Plot for LogNormal vs. Pareto

```julia
x_m = 1.0
x = range(1.0, 5, length = 100)
plot(log.(x),
     log.(1 .- cdf(Pareto(2.0, x_m), x));
     label = "Pareto",
     xlabel = L"\log(x)", size = (600, 400),
     legend = :bottomleft)
plot!(log.(x),
     log.(1 .- cdf(LogNormal(1.0, 0.5), x));
     label = "LogNormal")
```



Heavy Tailed Distributions

- See **Heavy-Tailed Distributions** by Stachurski and Sargent for more
- We previously looked at the LLN and Monte Carlo methods for calculating functions of a distribution from samples
- Crucial in these was a question of whether a particular distribution had a particular moment.
  - → e.g. for the Cauchy distribution, the mean does not even exist

Power-Law Tails

- The Pareto distribution is a special case of a power-law distribution
- A power-law distribution asymptomatically behaves like a Pareto distribution, with some α tail parameter. i.e. $\mathbb{P}(X > x) \propto x^{-\alpha}$ for large x
- More formally, there exists some c and some $\alpha > 0$ such that

$$\lim_{x \to \infty} x^{\alpha} \mathbb{P}(X > x) = c$$

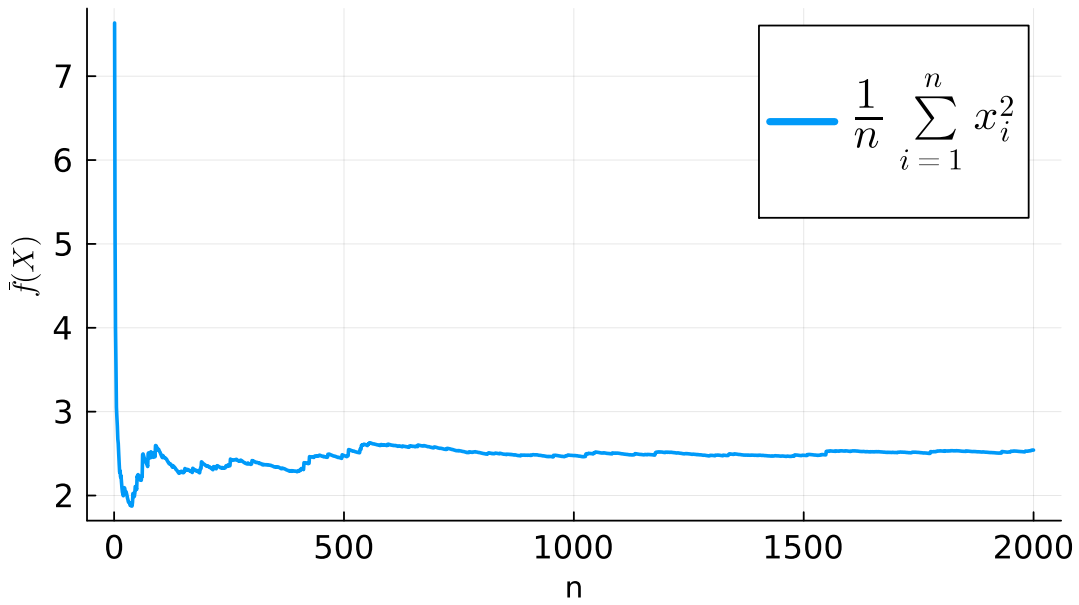
Failures of LLNs?

- For power-law tails, you may find that the not all moments exist
- In particular, for a Power-law distribution, there are only moments for $k < \alpha$
  - → For example, with $\alpha = 1$ the mean and variance don't exist
  - → For $\alpha = 1.8$ the mean exists, but the variance doesn't
- Of course, with finite data you will always be able to find a mean and variance, but with more data you may see them diverge

Example with Pareto and $\alpha = 3$

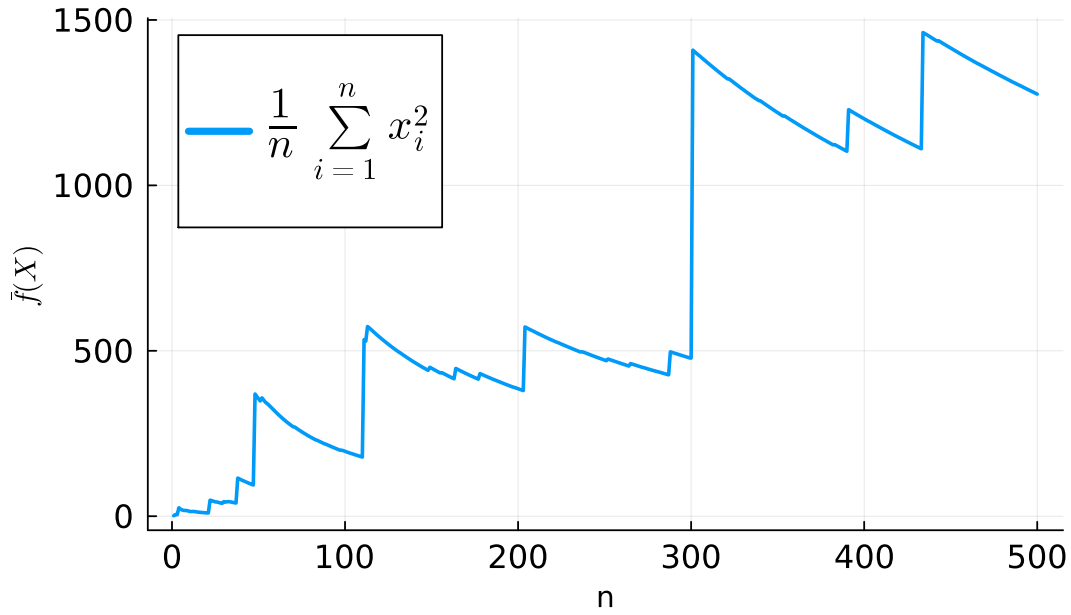
```julia
N = 2000
dist = Pareto(3.0, 1.0)
@show var(dist) + mean(dist)^2
x_draws = rand(dist, N)
f_x_draws = x_draws.^2
f_means = cumsum(f_x_draws)./(1:N)
plot(1:length(f_means), f_means;
     label=L"\frac{1}{n}\sum_{i=1}^n x_i^2",
     xlabel="n", ylabel=L"\bar{f}(X)",
     size=(600,400))
```

```
var(dist) + mean(dist) ^ 2 = 3.0
```



Example with Pareto and $\alpha = 1.0$

```
N = 500
dist = Pareto(1.0, 1.0)
x_draws = rand(dist, N)
f_x_draws = x_draws.^2
f_means = cumsum(f_x_draws)./(1:N)
plot(1:length(f_means), f_means;
     label=L"\frac{1}{n}\sum_{i=1}^n x_i^2",
     xlabel="n", ylabel=L"\bar{f}(X)",
     size=(600,400))
```



Empirical CDFs

- Given a pmf, $\boldsymbol{p}$, of a discrete-valued random variable with ordered values, we can do the CDF as $\boldsymbol{F(x_i) = \sum_{j=1}^{i} p(x_j)}$
- We can find an **empirical counterpart** for an unweighted vector $\{X_n\}_{n=1}^{N}$ of observations as

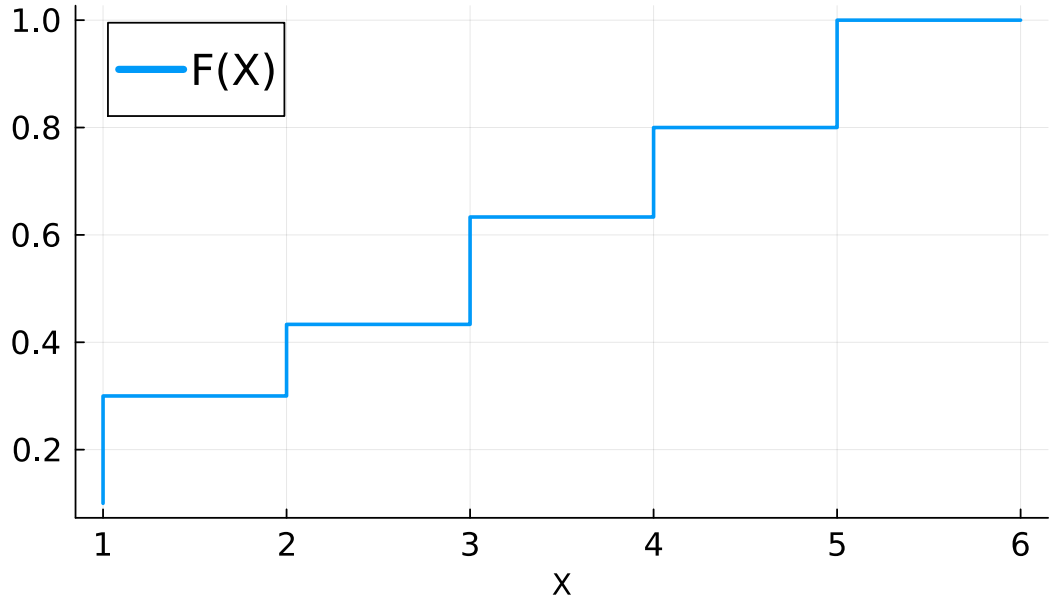
$$\hat{F}(x) = \frac{\text{number of observations } X_n \leq x}{N}$$

- With the equivalent CCDF as $1 - \hat{F}(x)$.

Empirical CDF with Discrete Number of Values

- If there are a finite number of values, count the observations

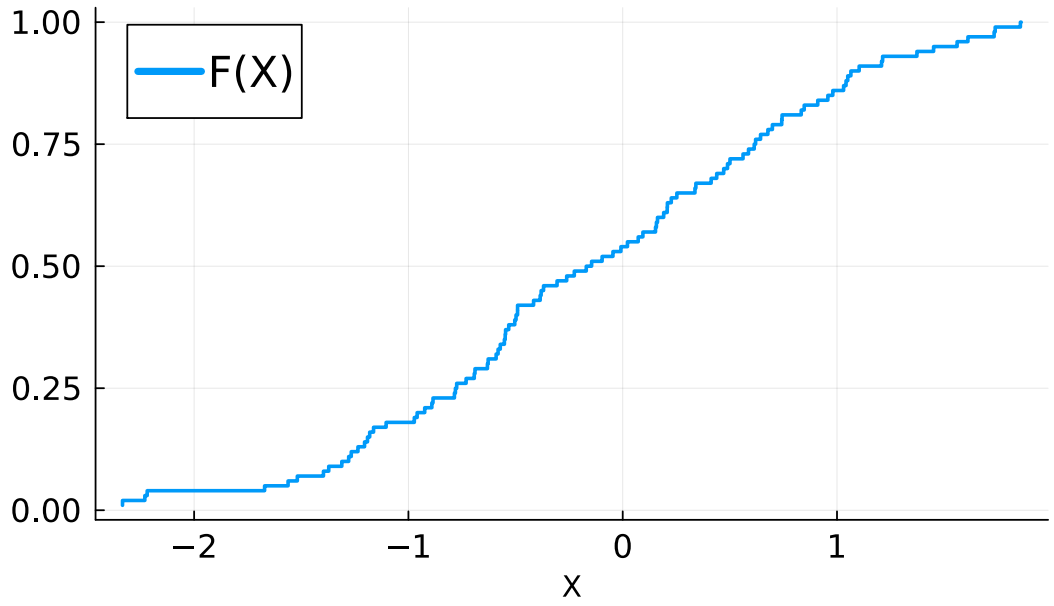
```
X_data = sort(rand(1:6, 30))
X = unique(X_data)
counts = [sum(X_data .== x) for x in X]
cumulative_counts = cumsum(counts)
F = cumulative_counts / length(X_data)
plot(X, F; xlabel="X", label = "F(X)",
     seriestype = :steppre,
     size = (600, 400))
```



Empirical CDF from Continuous Data

- In cases where the data is continuous counting is just sorting

```
n = 100
X_data = randn(n)
F = (1:n) / n
plot(sort(X_data), F; xlabel="X",
     label = "F(X)", seriestype = :steppre,
     size = (600, 400))
```



(Crude) Tail Parameter Estimation

- The tail parameter α is the negative slope of the log-log plot of the CCDF
- Hence, we use a linear regression. Tail parameter $\alpha = -a$

$$\underbrace{\log(1 - F_i)}_{\equiv y_i} = b + a\underbrace{\log(X_i)}_{\equiv x_i} + \epsilon_i$$

  → Run regression with package (e.g. GLM.jl) or manually

- Discrete data: CCDF of last point is zero, cannot take a log

  → One solution is just to drop that last point in the regression

See Rank — 1/2: A Simple Way to Improve the OLS Estimation of Tail Exponents by Gabaix and Ibragimov for a more sophisticated approach

Calculation of Coefficients

- Given the least squares problem

$$\min_{a,b} \sum_{i=1}^{n} (y_i - (ax_i + b))^2$$

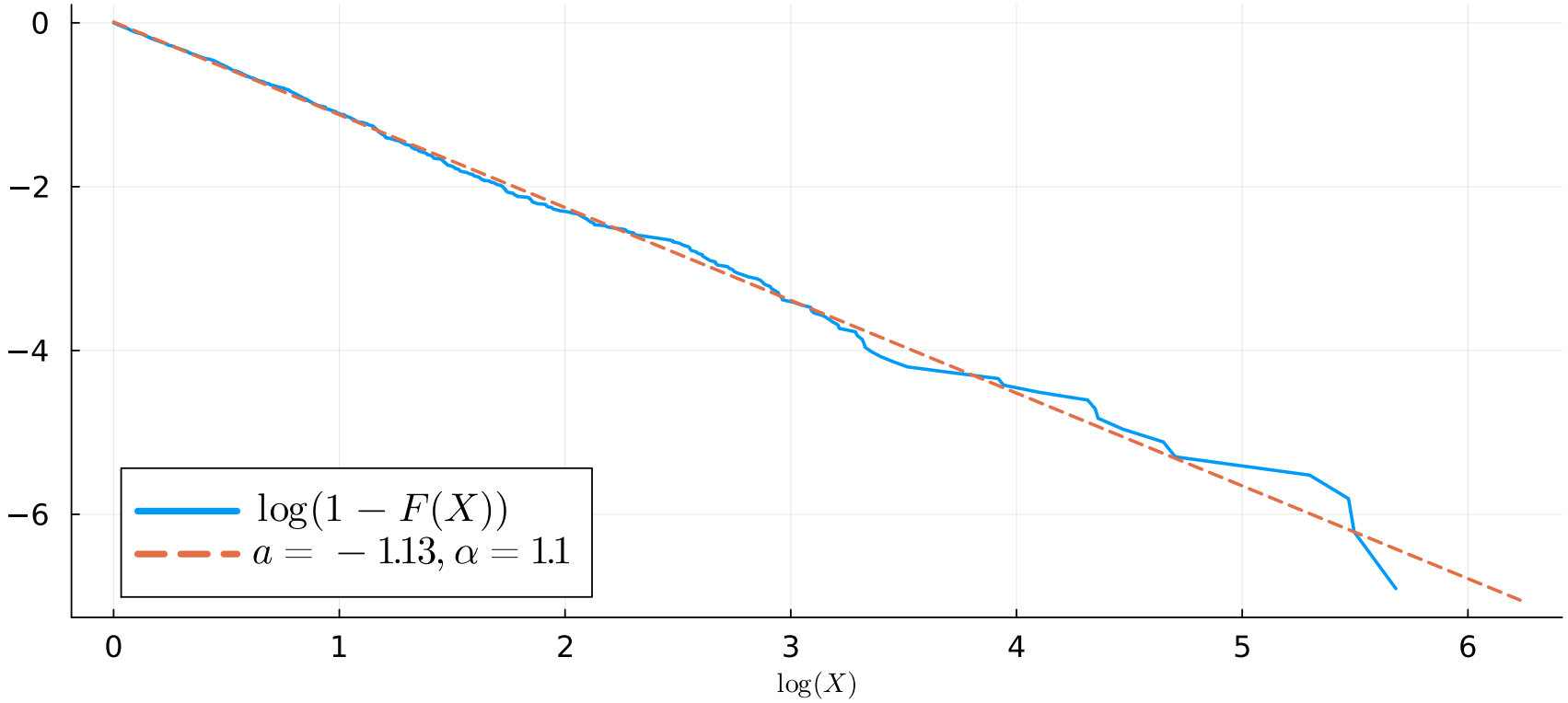
- Calculate the means, $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ and $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$
- Then the coefficients are

$$a = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}, \quad b = \bar{y} - a\bar{x}$$

Example with Pareto and $\alpha = 1.1$

```
function simple_regression(x, y)
  x_bar = mean(x)
  y_bar = mean(y)
  a = sum((x .- x_bar).*(y .- y_bar)) / sum((x .- x_bar).^2)
  b = y_bar - a * x_bar
  return (;a, b)
end
N = 1000
alpha = 1.1
X = sort(rand(Pareto(alpha, 1.0), N))
F = (1:N) / N
y = log.(1 .- F)
x = log.(X)
(;a, b) = simple_regression(x[1:end-1], y[1:end-1])
plot(x, y; label = L"\log(1-F(X))", xlabel=L"\log(X)")
plot!(x, a*x .+ b; label = L"a=%$(round(a, digits = 2)), \alpha = %$alpha", style = :dash)
```

Example with Pareto and $\alpha = 1.1$

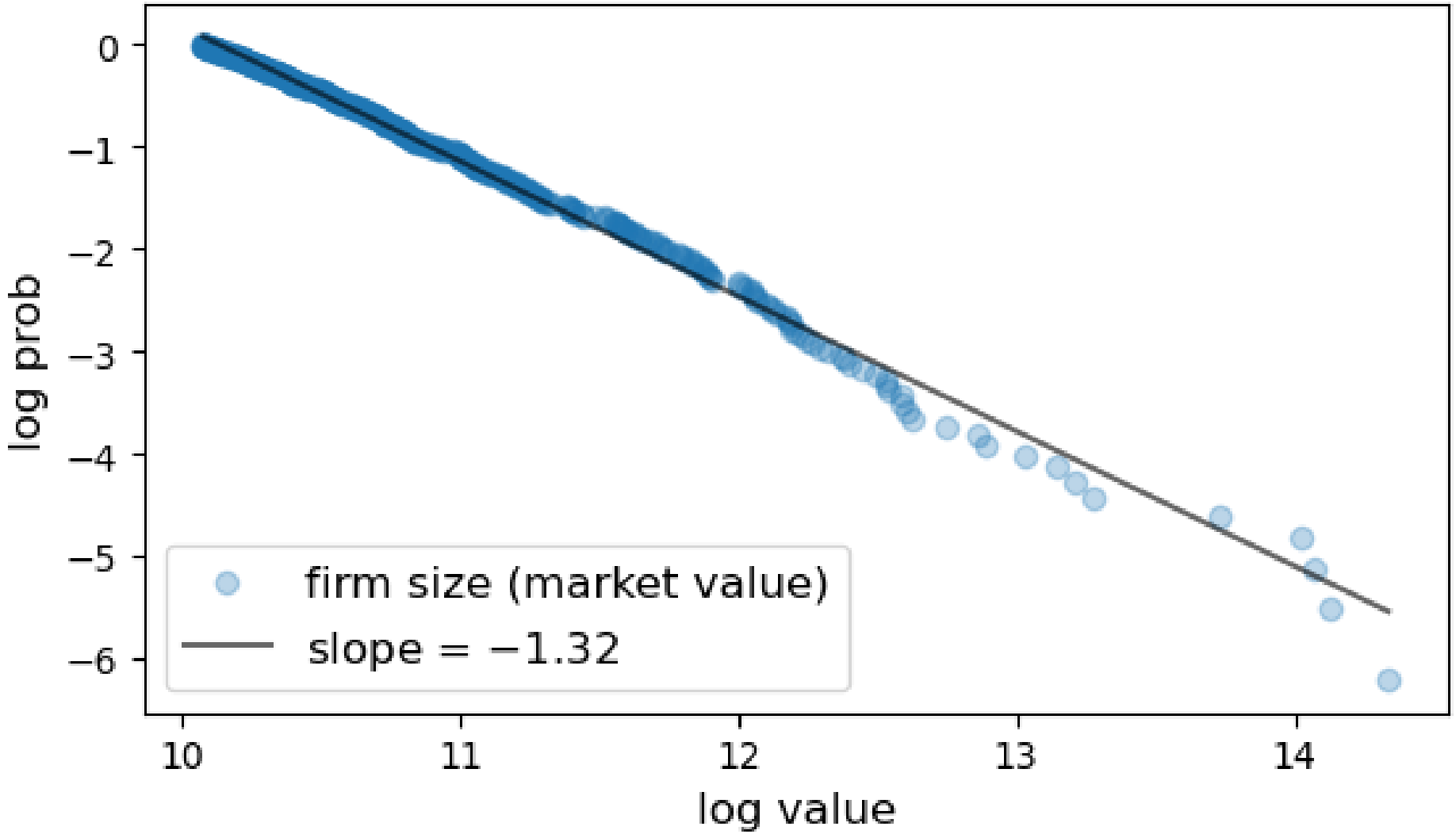


Empirical Tails

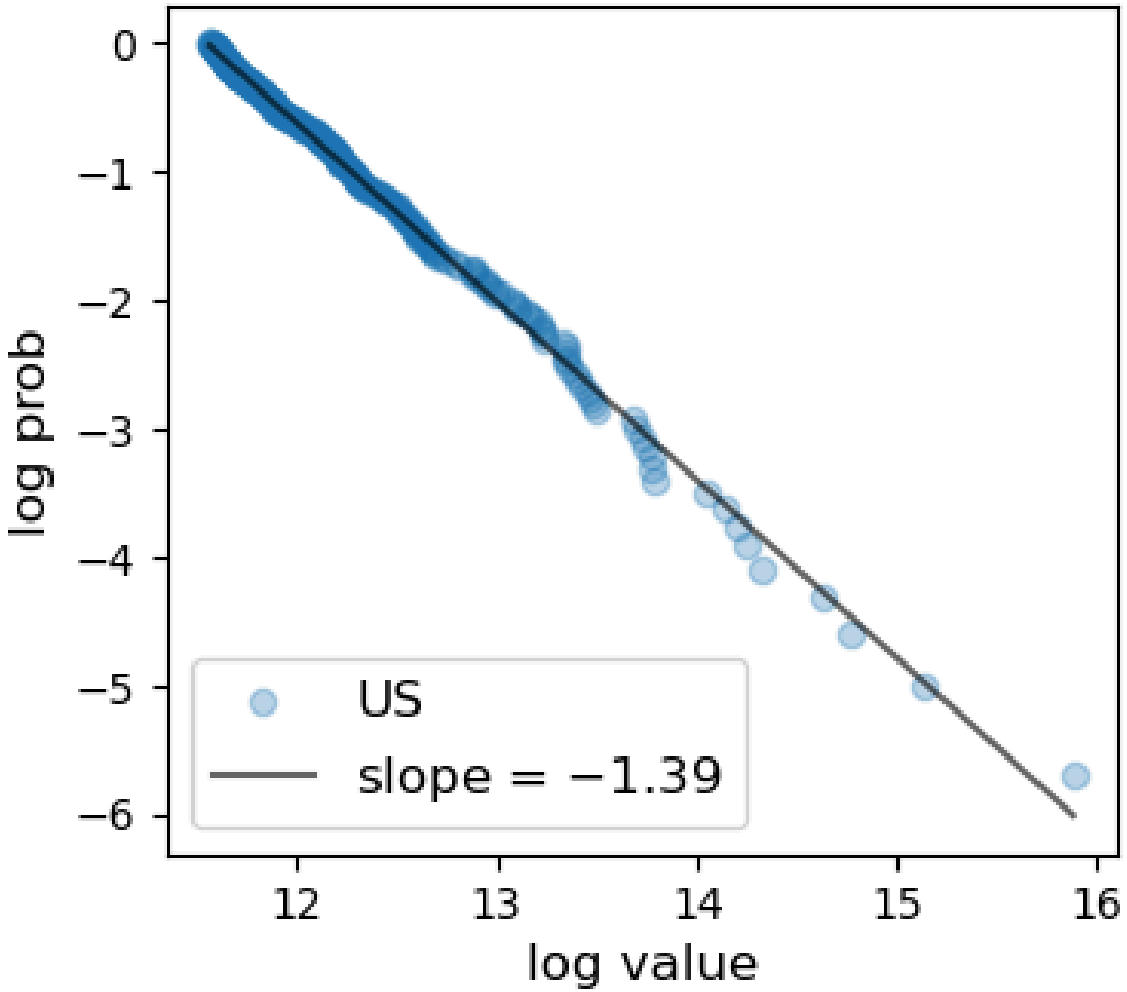
Empirical Evidence of Tails

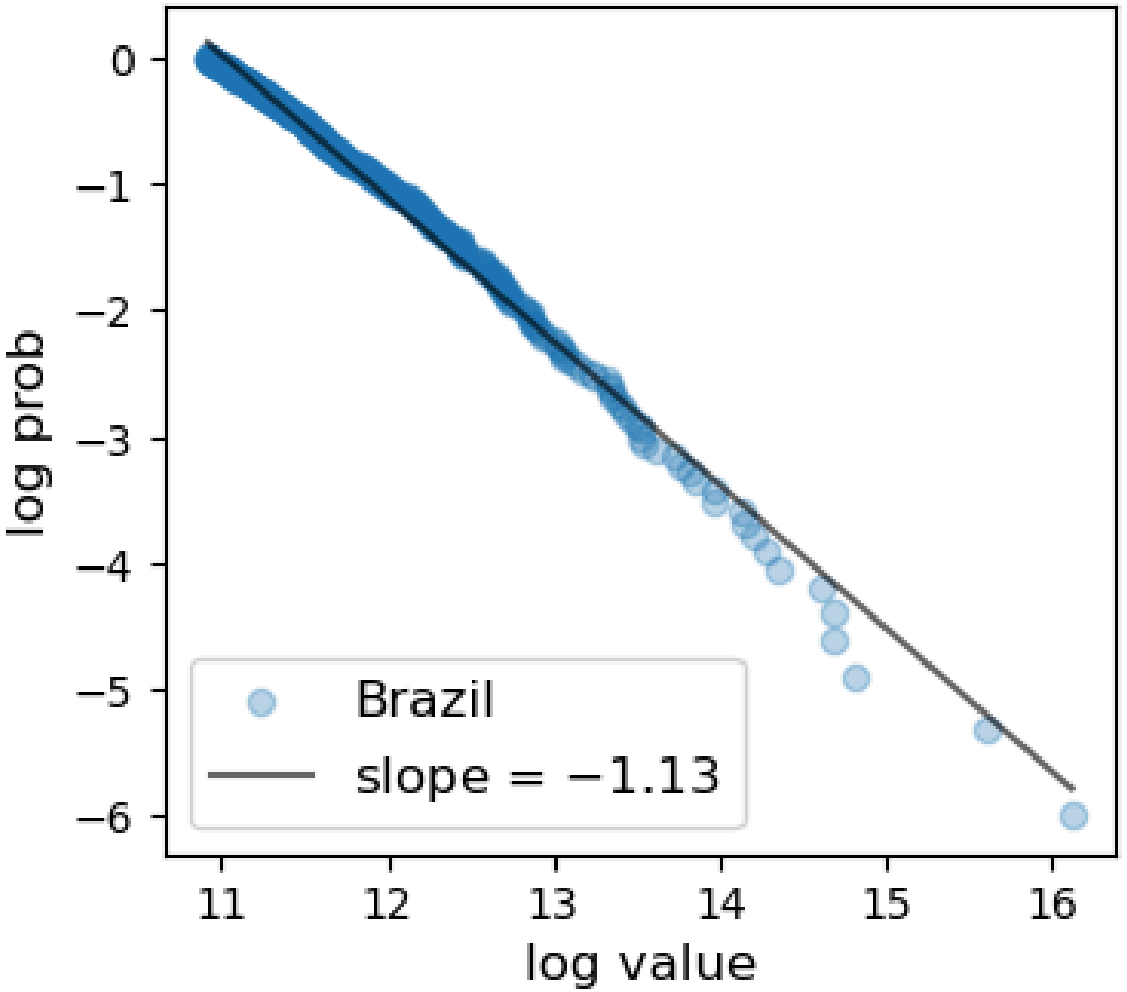
- It is sometimes difficult, with finite data, to distinguish between a heavy-tailed distribution and a then-tailed one
- The issue is that in either case there are typically not that many observations with large values, even if there are more for power-laws.
- Some classic examples of power-law tails in the data
  - → All from **Heavy-Tailed Distributions** by Stachurski and Sargent

Largest 500 firms in 2020 taken from Forbes Global 2000

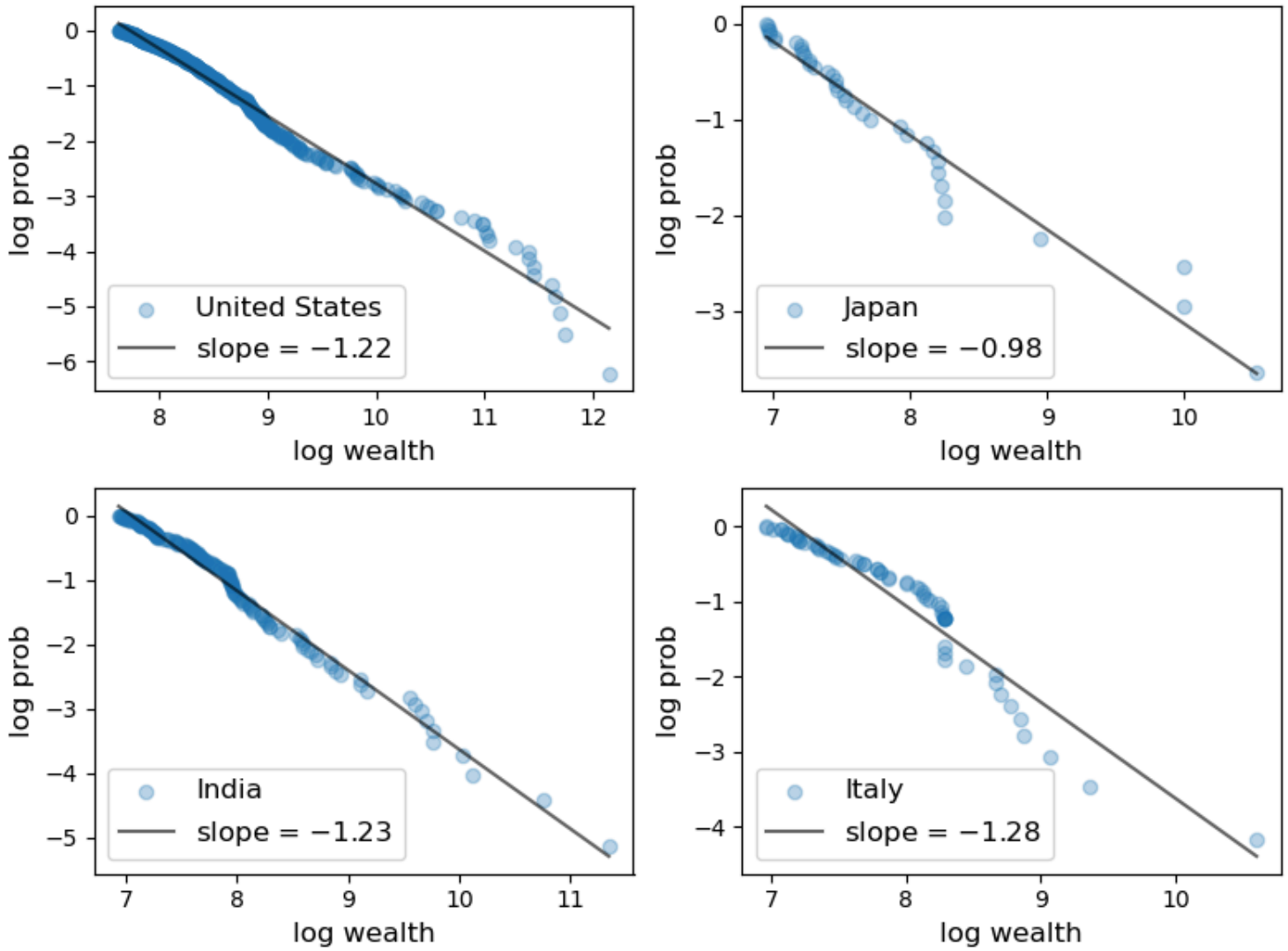


City Sizes in the US and Brazil

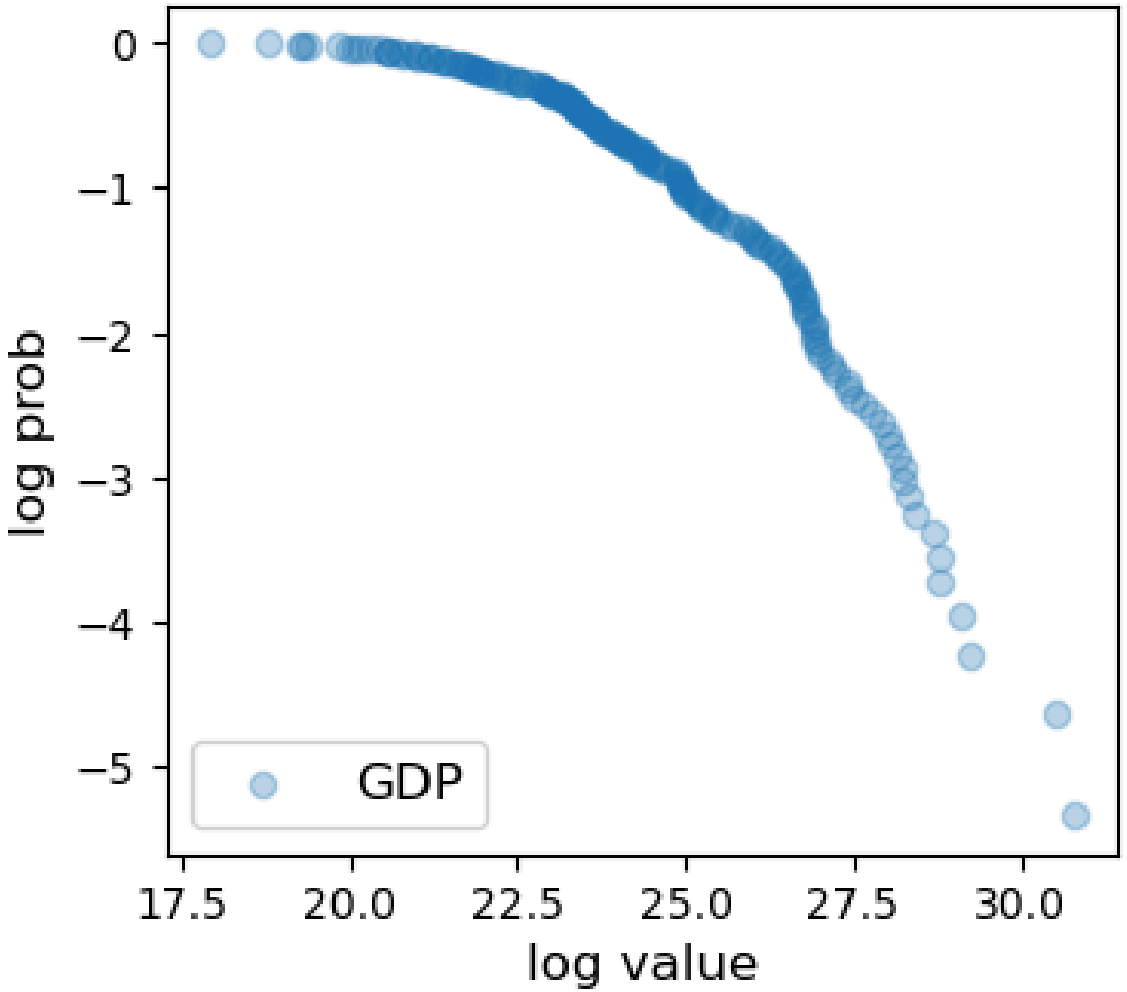


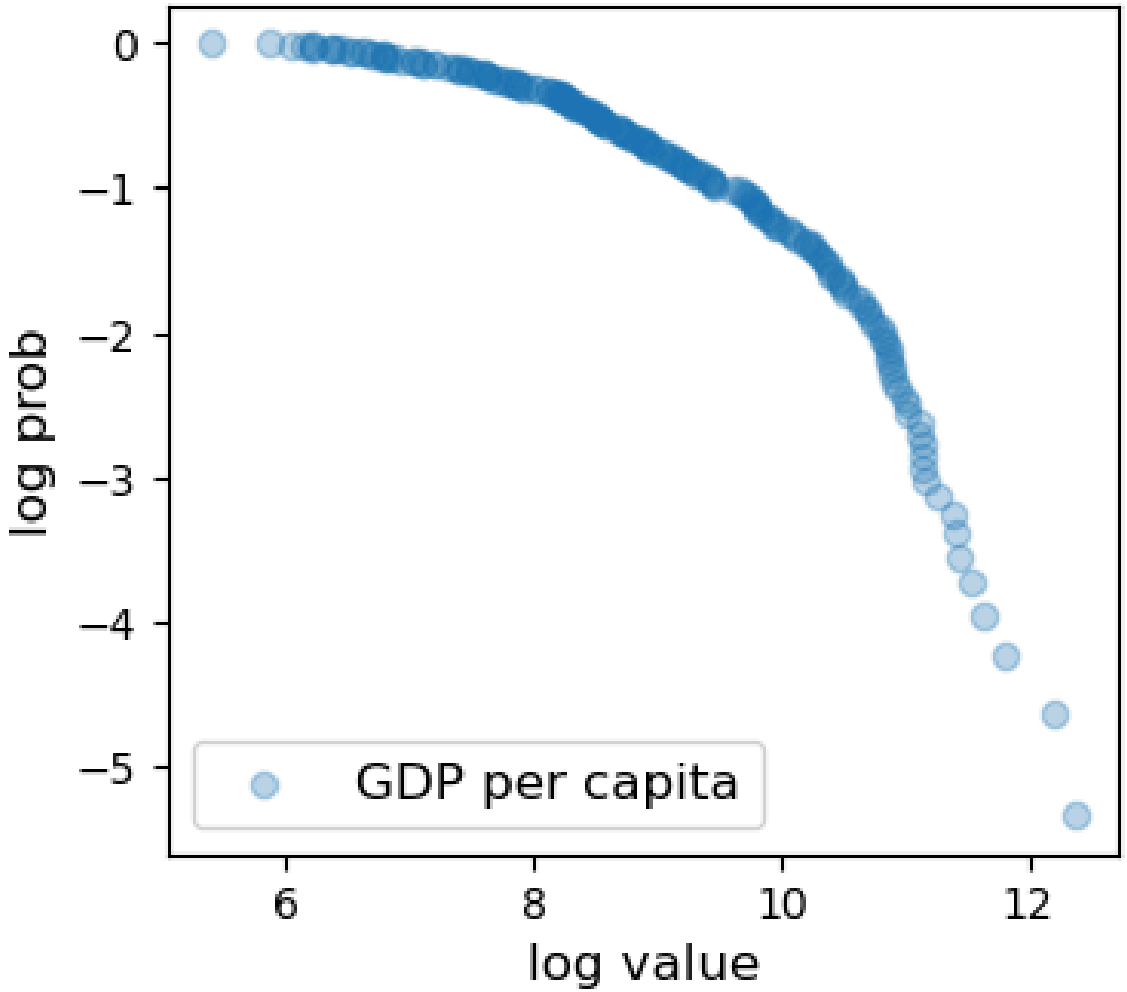


Wealth Distribution Across Countries



GDP Across Countries





Firm Dynamics

Multiplicative Dynamics and Gibrat's Law

- **Gibrat's Law** is a simple model of firm dynamics which proposes that the growth rate of a firm is independent of its size
  - → Note, this is a "law" on the stochastic process, not the stationary distribution that comes out of it
  - → Can show that a proportional growth process will lead to a lognormal distribution of firm sizes over time
- Does it hold in the data?
- Not exactly, even if a good starting point (e.g.,**Gibrat's Legacy** by John Sutton)
  - → Small firms tend to grow faster than large firms
  - → Volatility is higher for small firms
  - → Rather than lognormal, seems closer to Zipfs law

Zipf Distribution? $\alpha = 1.059$

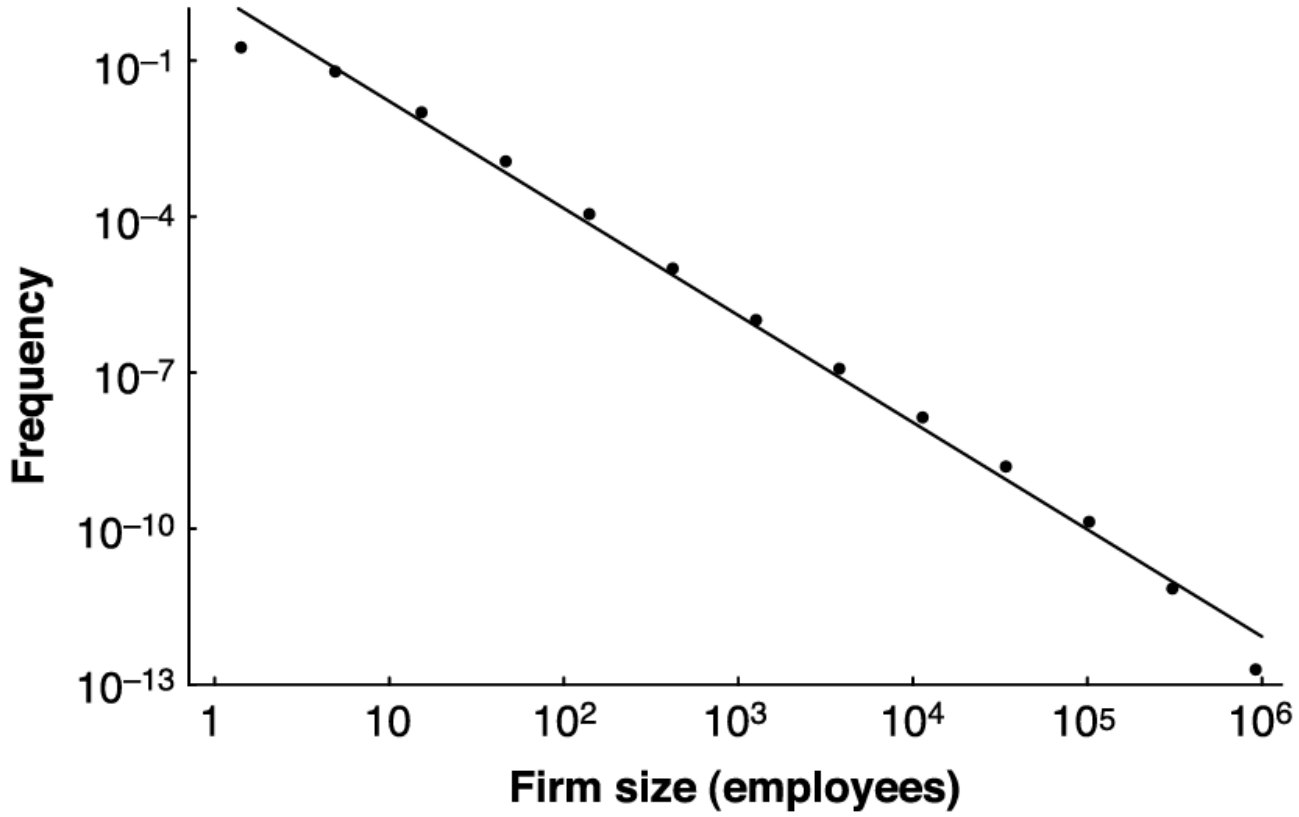


Fig. 1. Histogram of U.S. firm sizes, by employees. Data are for 1997 from the U.S. Census Bureau, tabulated in bins having width increasing in powers of three (*30*). The solid line is the OLS regression line through the data, and it has a slope of 2.059 (SE = 0.054; adjusted $R^2 = 0.992$), meaning that $\alpha = 1.059$; maximum likelihood and nonparametric methods yield similar results. The data are slightly concave to the origin in log-log coordinates, reflecting finite size cutoffs at the limits of very small and very large firms.

See https://www.science.org/doi/10.1126/science.1062081 by Axtell

Which of Stochastic Processes Lead to Power Laws?

- Zipf's Law is a type of distribution: The size of the nth largest is inversely proportional to n
  - → Equivalent to a discrete version of a Pareto distribution with $\alpha = 1$
- Power Laws in Economics and Finance by Xavier Gabaix is a good reference on Power laws and where they come from
- The key: multiplicative growth processes + some distortion at the bottom of the distribution
  - → bankruptcy
  - → exit and entry
  - → additive shocks which distort the bottom disproportionately

Reminder on Kesten Processes

- Recall the **Kesten Process**, which generalizes this by adding in the y_{t+1} term

$$X_{t+1} = a_{t+1} X_t + y_{t+1}$$

  → a_{t+1} is an IID growth rate, y_{t+1} is an IID additive shock

- Another example of a related process

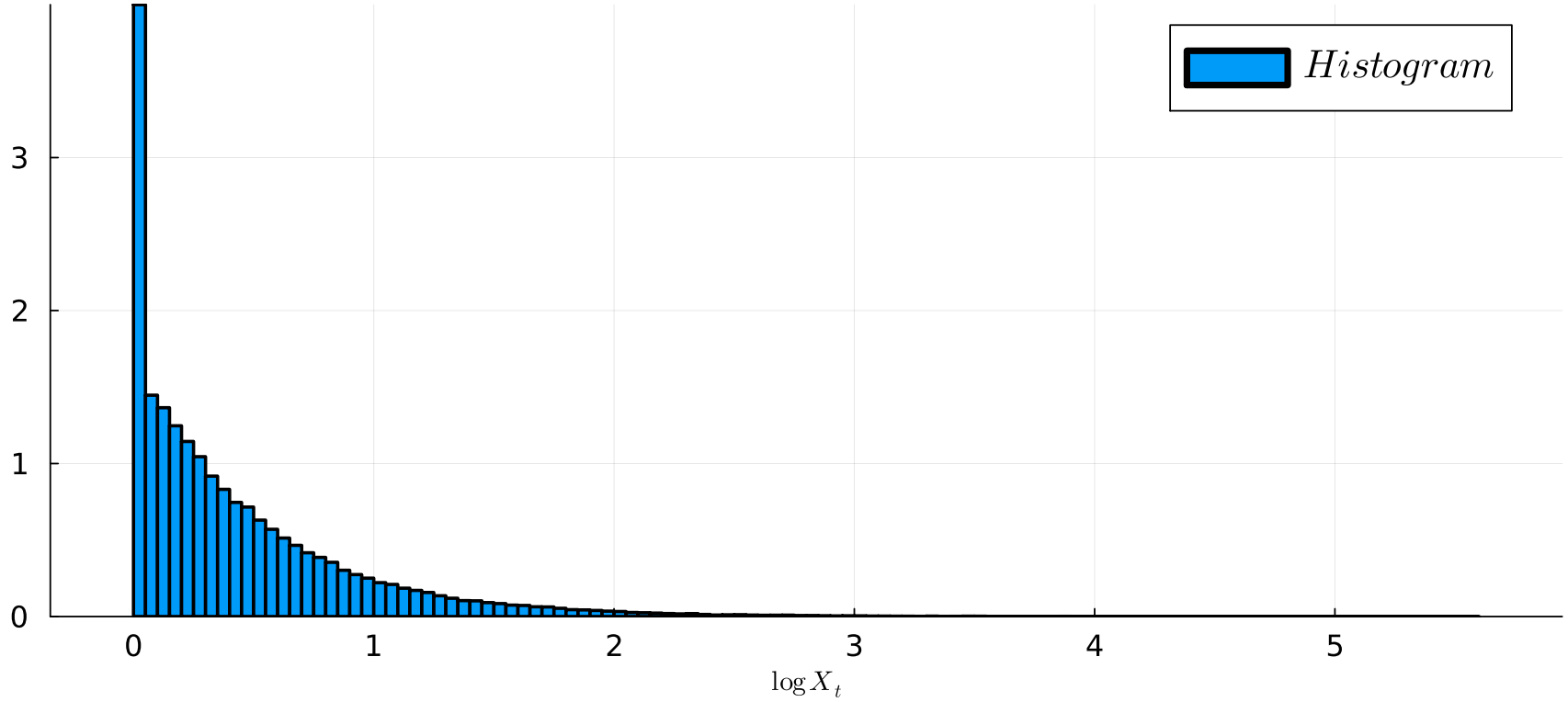
$$X_{t+1} = \max\{s_0, a_{t+1} X_t\}$$

Simulation of a Process for Growth

- Let firm size $X_{t+1} = \max\{s_0, a_{t+1} X_t\}$, where $\log a_{t+1} \sim \mathcal{N}(\mu, \sigma^2)$

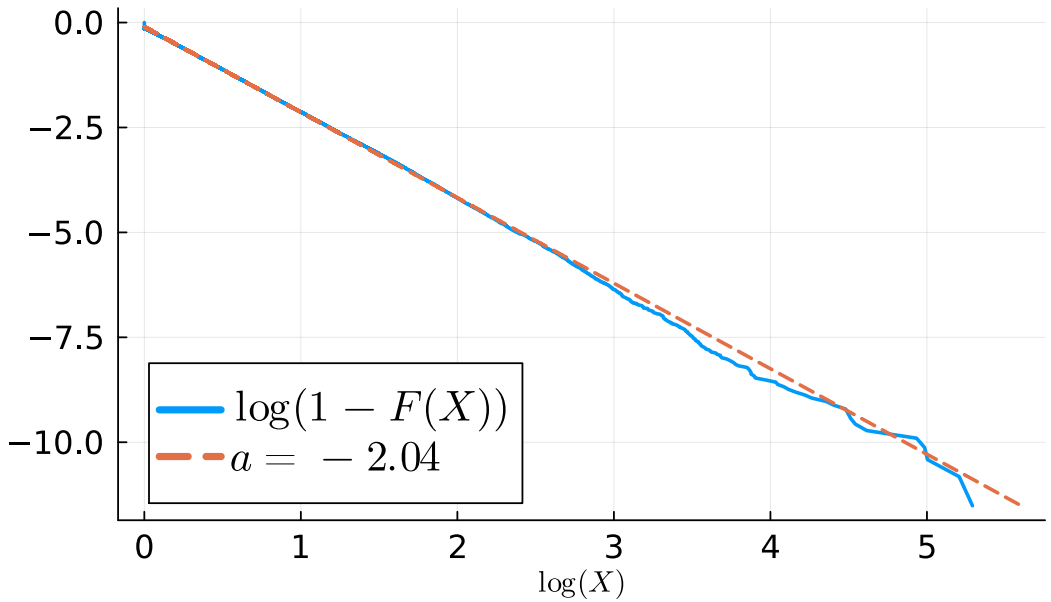
```
function iterate_map_iid_ensemble(f, dist, x0, T, num_samples)
    x = zeros(num_samples, T + 1)
    x[:, 1] .= x0
    for t in 2:(T + 1)
        x[:, t] .= f.(x[:, t - 1], rand(dist, num_samples))
    end
    return x
end
num_samples = 100000
T = 500
s_0 = 1.0 # exit/entry at X = 1.0
X_0 = 1.0
a_dist = LogNormal(-0.01, 0.1)
h(X, a) = max(s_0, a * X)
X = iterate_map_iid_ensemble(h, a_dist, X_0, T, num_samples)
histogram(log.(X[:, end]); label = L"Histogram", xlabel = L"\log X_t", normalized=true)
```

Simulation of a Process for Growth



Log-Log Plot

```
X_T = sort(X[:, end])
x_T = log.(X_T)
F_T = (1:length(X_T)) / length(X_T)
y_T = log.(1 .- F_T)
(;a, b) = simple_regression(x_T[1:end-1],
                            y_T[1:end-1])
plot(x_T, y_T; label = L"\log(1-F(X))",
     xlabel=L"\log(X)", size = (600, 400))
a_r = round(a, digits = 2)
plot!(x_T, a*x_T .+ b;
      label = L"a=%$a_r", style = :dash)
```



Lorenz Curves and Gini Coefficients

Visualizing Inequality

- Tails are helpful for seeing how inequality is distributed at the top-end (e.g., the top 1% or 0.1%)
  - → They can also help us understand processes which might generate inequality
  - → For example, Kesten processes, which we will come back to, are a class of processes which generate power-law tails
  - → This can influence tax policy/etc.
- However, the tail behavior is not very useful to understand inequality in the lower parts of the distribution

Lorenz Curves

- One popular graphical measure of inequality is the Lorenz curve.
- For a continuous distribution with pdf $f(x)$, cdf $F(x)$, and quantile $x = F^{-1}(p) \equiv Q(p)$

$$L(p) = \frac{\int_{-\infty}^{Q(p)} x f(x) dx}{\int_{-\infty}^{\infty} x f(x) dx}$$

- Which can be rewritten as $L(p) = \frac{\int_0^p Q(s) ds}{\int_0^1 Q(s) ds}$
- Intuition: the proportion of the population with less than p of the total income has $L(p)$ of the total income

Lorenz Curve Given Data

- In the case of **unweighted discrete data** we have a simple empirical version of the Lorenz curve. **Weighted** versions are useful if you bin data (e.g. quintiles)
- Given **sorted** $v_1, \ldots v_n$, we find the empirical CDF (previous slides) is $F(v_i) \equiv F_i \equiv \frac{i}{n}$
- The Lorenz curve is

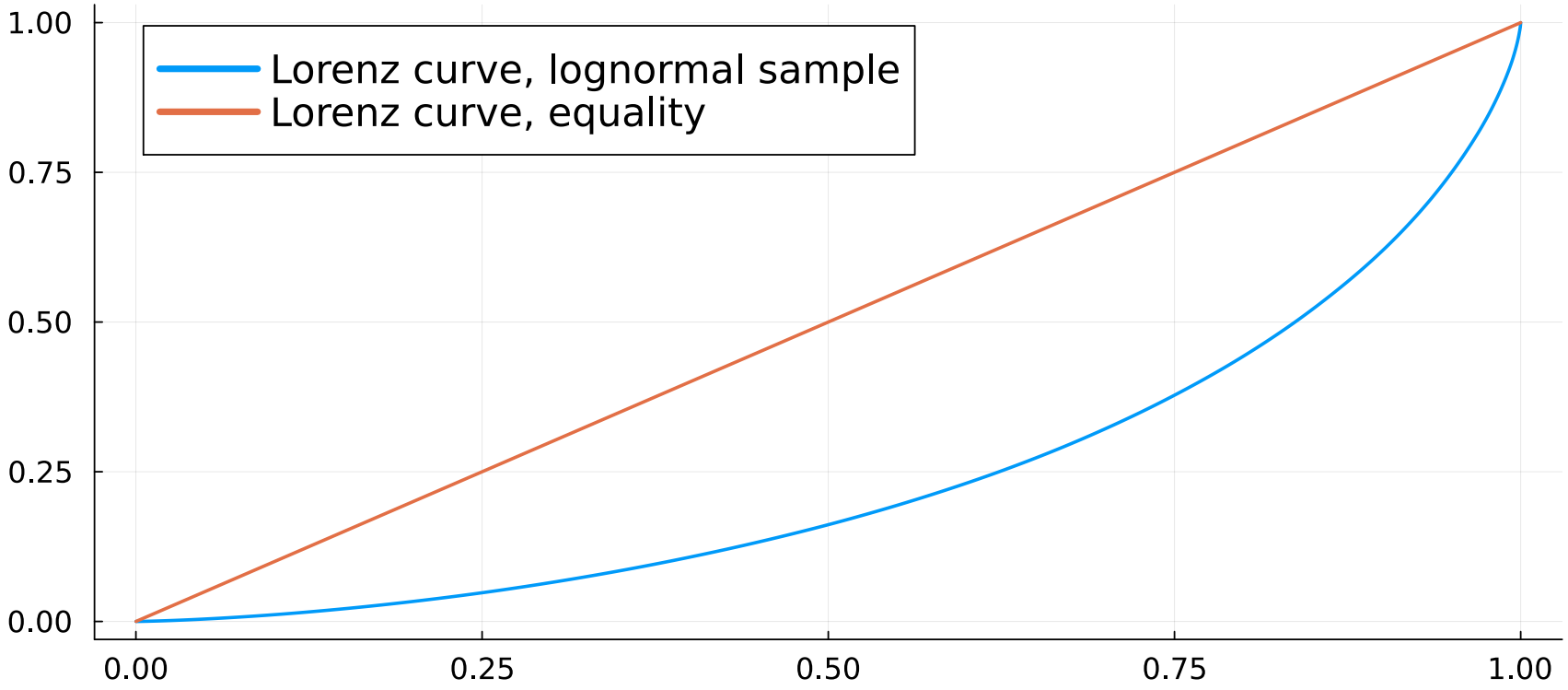
$$S_i = \frac{1}{n} \sum_{j=1}^{i} v_j$$

$$L_i = \frac{S_i}{S_n}$$

Implementation

```
function lorenz(v)  # assumed sorted vector
    S = cumsum(v)  # cumulative sums: [v[1], v[1] + v[2], ... ]
    F = (1:length(v)) / length(v) # empirical CDF since everyone has the same weight!
    L = S ./ S[end]
    return (; F, L) # returns named tuple
end
n = 10_000
w = sort(exp.(randn(n)));  # lognormal draws
(; F, L) = lorenz(w)
plot(F, L, label = "Lorenz curve, lognormal sample", legend = :topleft)
plot!(F, F, label = "Lorenz curve, equality")
```

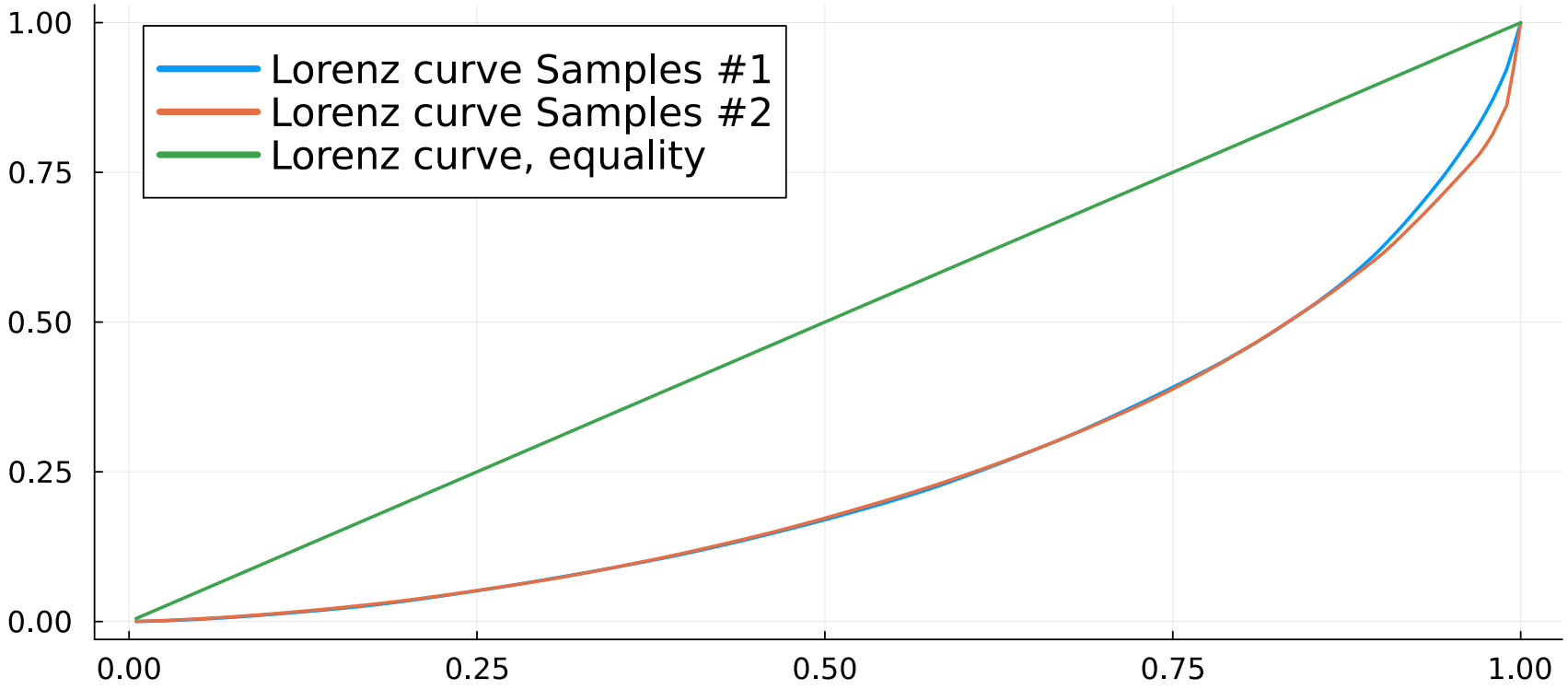
Implementation



With Cruder Samples Still Fairly Smooth

```
n = 200
w = sort(exp.(randn(n)));  # lognormal draws
(; F, L) = lorenz(w)
plot(F, L, label = "Lorenz curve Samples #1", legend = :topleft)
w = sort(exp.(randn(n)));  # lognormal draws
(; F, L) = lorenz(w)
plot!(F, L, label = "Lorenz curve Samples #2")
plot!(F, F, label = "Lorenz curve, equality")
```

With Cruder Samples Still Fairly Smooth



Interpretation

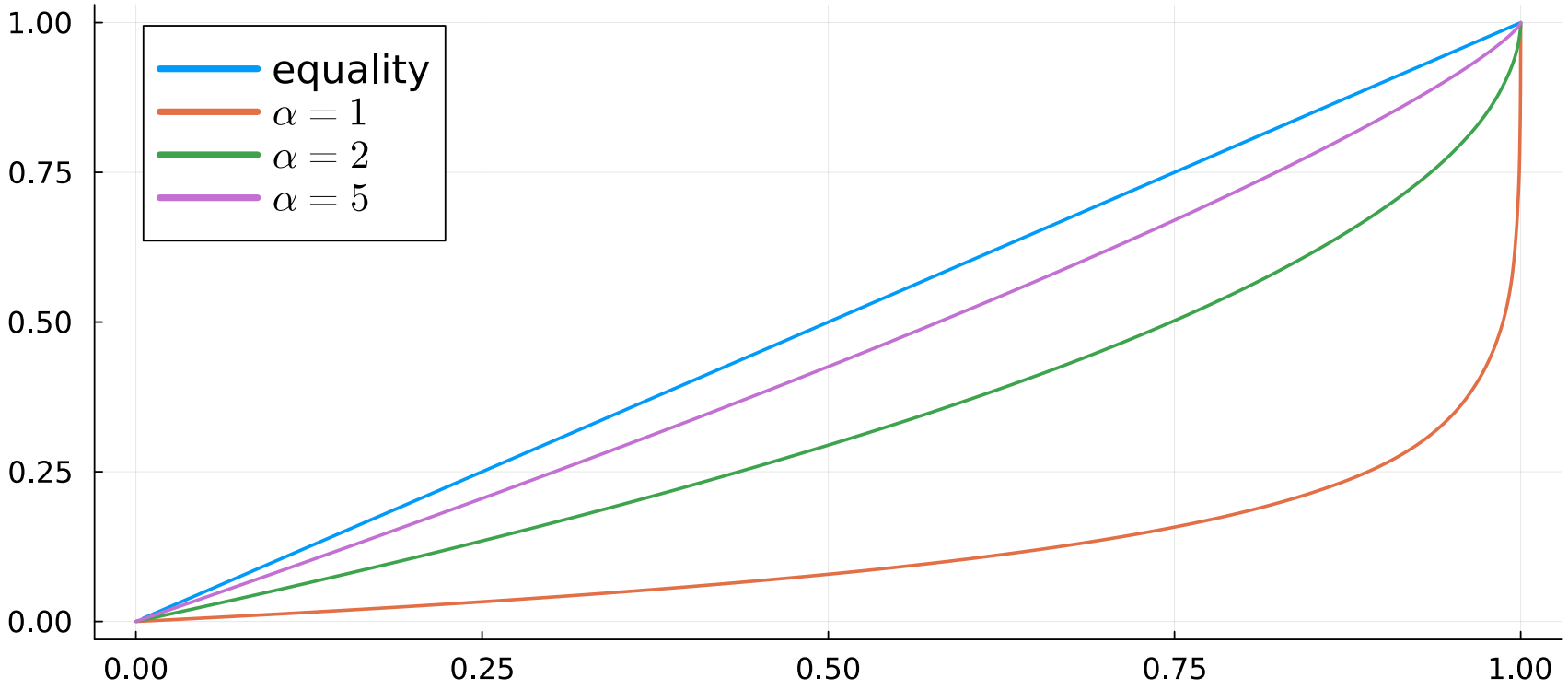
- if point (x, y) lies on the curve, it means that, collectively, the bottom $(100x)\%$ of the population holds $(100y)\%$ of the wealth.
- The “equality” line is the 45 degree line, i.e. the Lorenz curve under perfect equality.
- In this example, the bottom 80% of the population holds around 40% of total wealth.

Lorenz Curve for Pareto

- Can verify analytically that $L(p) = 1 - (1-p)^{1-1/\alpha}$ for the Pareto distribution

```
a_vals = (1, 2, 5)
n = 10_000
plt = plot(F, F, label = "equality", legend = :topleft)
for a in a_vals
    u = rand(n)
    y = sort(rand(Pareto(a, 1.0), n))
    (; F, L) = lorenz(y)
    plot!(plt, F, L, label = L"\alpha = %$a")
end
plt
```

Lorenz Curve for Pareto



Gini Coefficients

- The **Gini Coefficient** is a summary measure of the Lorenz curve
  - → Integral between the Lorenz curve and the line of equality
  - → Gini is zero with no inequality, and one if concentrated in single individual
- With sorted, unweighted discrete set $\{v_1, \ldots v_n\}$ there is a **simplification**

$$G = \frac{2\sum_{i=1}^{n} iv_i}{n\sum_{i=1}^{n} v_i} - \frac{n+1}{n}$$

Calculation and Comparison to Theoretical

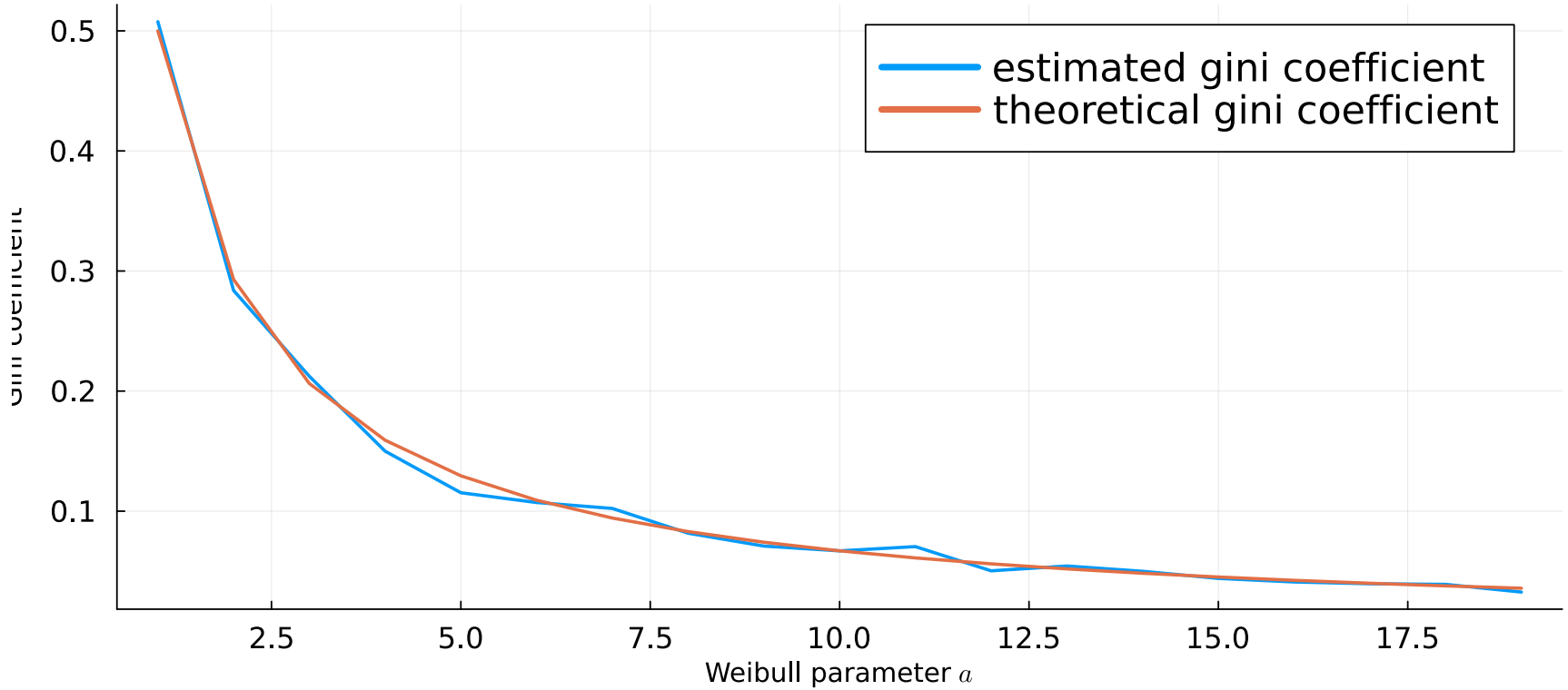
- The Weibull distribution, $f(x) = ax^{a-1}e^{-x^a}$, has a Gini coefficient of $1 - 2^{1/a}$

```
function gini(v)
    return (2 * sum(i * y for (i, y) in enumerate(v)) / sum(v)
           - (length(v) + 1)) / length(v)
end

a_vals = 1:19
n = 100
ginis = [gini(sort(rand(Weibull(a), n))) for a in a_vals]
ginis_theoretical = [1 - 2^(-1 / a) for a in a_vals]

plot(a_vals, ginis, label = "estimated gini coefficient",
     xlabel = L"Weibull parameter $a$", ylabel = "Gini coefficient")
plot!(a_vals, ginis_theoretical, label = "theoretical gini coefficient")
```

Calculation and Comparison to Theoretical



Wealth and Income Distribution

Income and Wealth

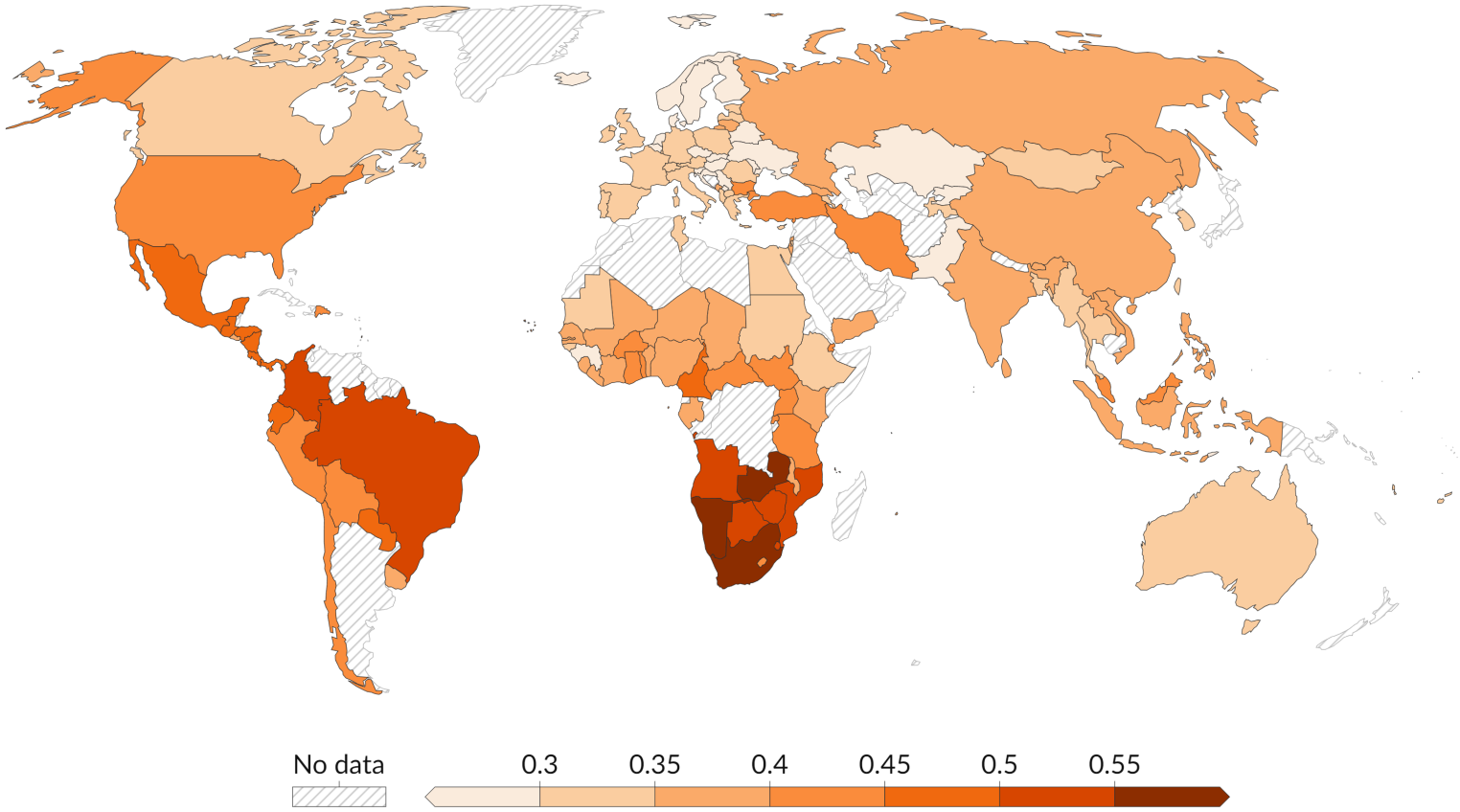
- The degree of inequality in income and wealth can be very different
  - → Income is a flow variable, wealth is a stock variable
  - → Income is taxed progressively
  - → Wealth is taxed proportionally in order to avoid double taxation (i.e., in principle, income was already taxed before purchasing assets)
  - → Different rates of return for different types of assets
  - → Inheritance of wealth vs. human capital
- Some data sources: World Inequality Database, Our World in Data, Stone Foundation and many OECD/Fed sources
- Theory Survey: Skewed Wealth Distributions: Theory and Empirics by Benhabib and Bisin

Measurement Issues

- Measurement and interpretation is tricky for both
- Panel data used by economists (e.g. PSID and Survey of Consumer Finances), but have difficulty sampling the rich
- Administrative data (e.g., social security records) help for income but not so much for wealth
- In many economies, a significant portion of wealth is in social security promises for many people.
  - → e.g., if we taxed everyone at 90% and used all of that for public pensions, then only calculating wealth from assets would be misleading
  - → How to handle the "zeros"? Maybe gini isn't ideal for wealth
- Mapping from wealth/income to consumption (and ultimately welfare)?

Income inequality: Gini coefficient, 2019

The Gini coefficient[1] measures inequality on a scale from 0 to 1. Higher values indicate higher inequality. Depending on the country and year, the data relates to income measured after taxes and benefits, or to consumption, per capita[2].



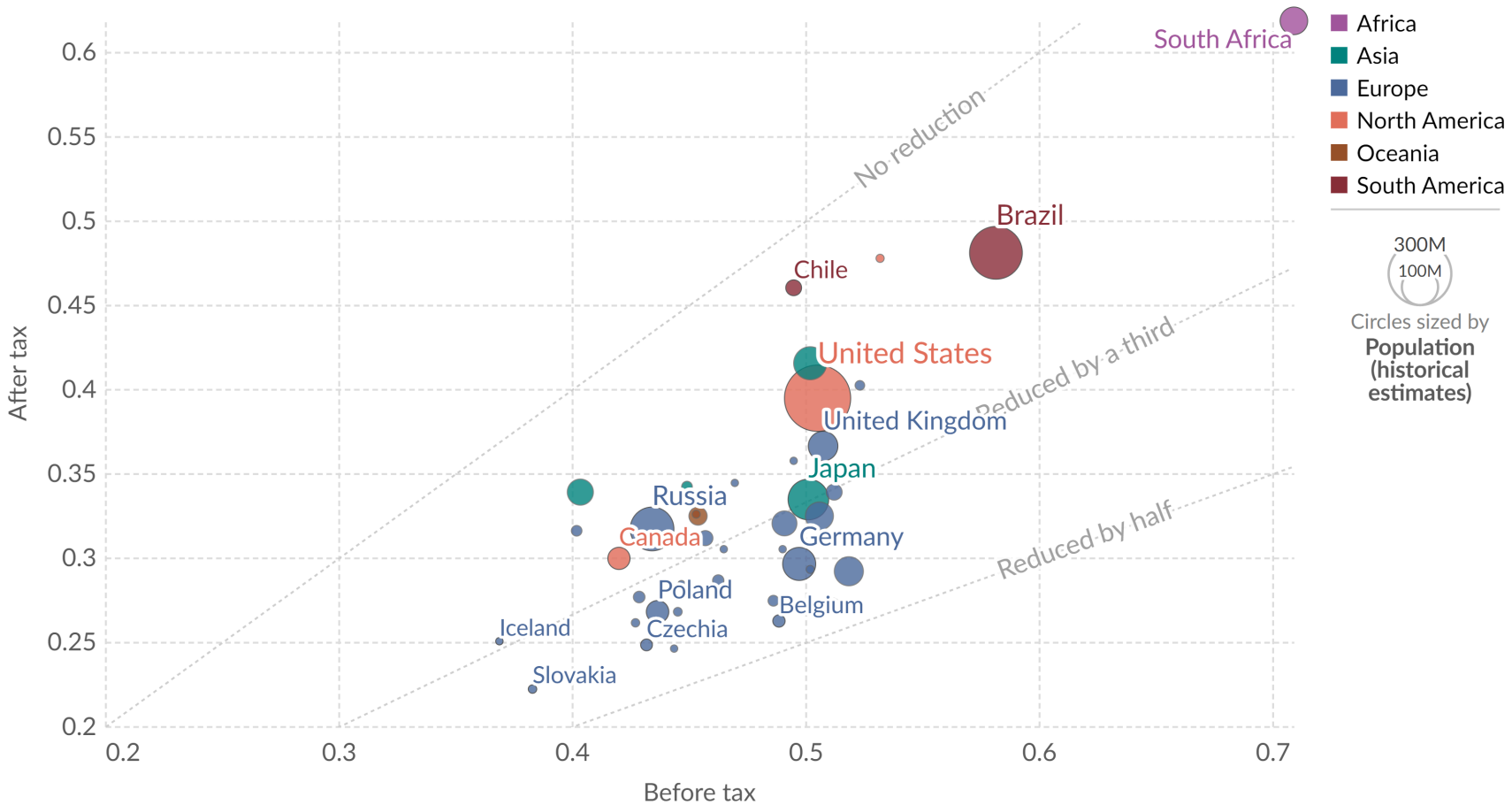
Data source: World Bank Poverty and Inequality Platform (2023)

OurWorldInData.org/economic-inequality | CC BY

Note: Income and consumption estimates are available separately in this Data Explorer.

Income inequality: Gini coefficient before and after tax, 2019

Inequality is measured here in terms of the Gini coefficient[1] of income before taxes on the x-axis, and after taxes on the y-axis.



Data source: OECD Income Distribution Database (2023)
Note: Income has been equivalized[2].

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From From Benhabib, Bisin, Luo

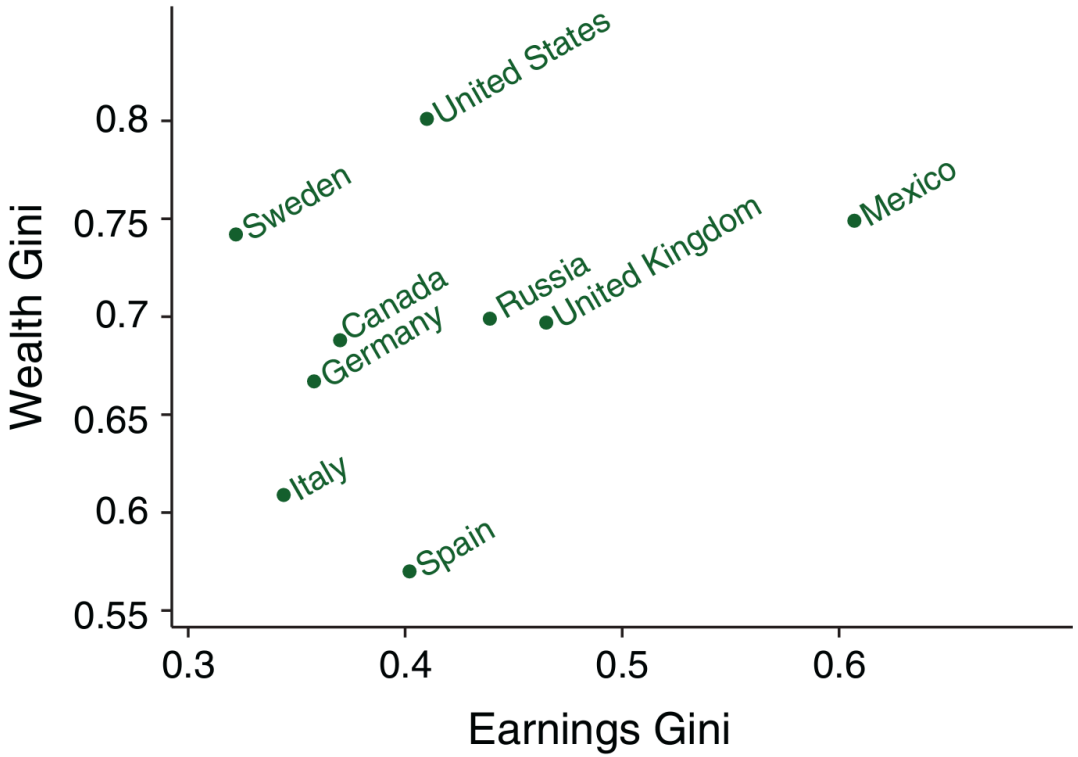


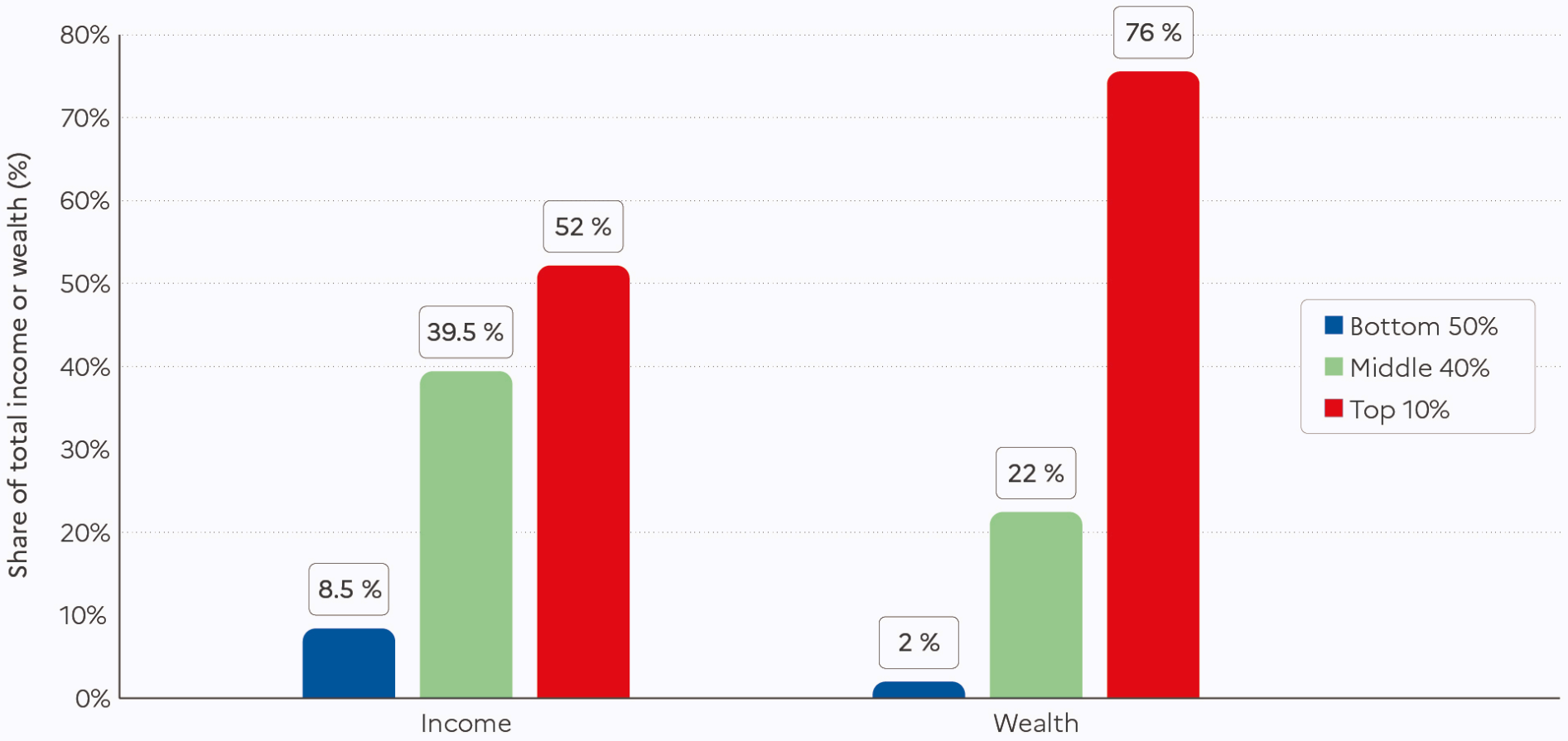
FIGURE 1. EARNINGS AND WEALTH GINI

Sources: Wealth: Davies et al. (2011). Earnings: (Krueger et al. 2010).

Cross-Country Comparisons

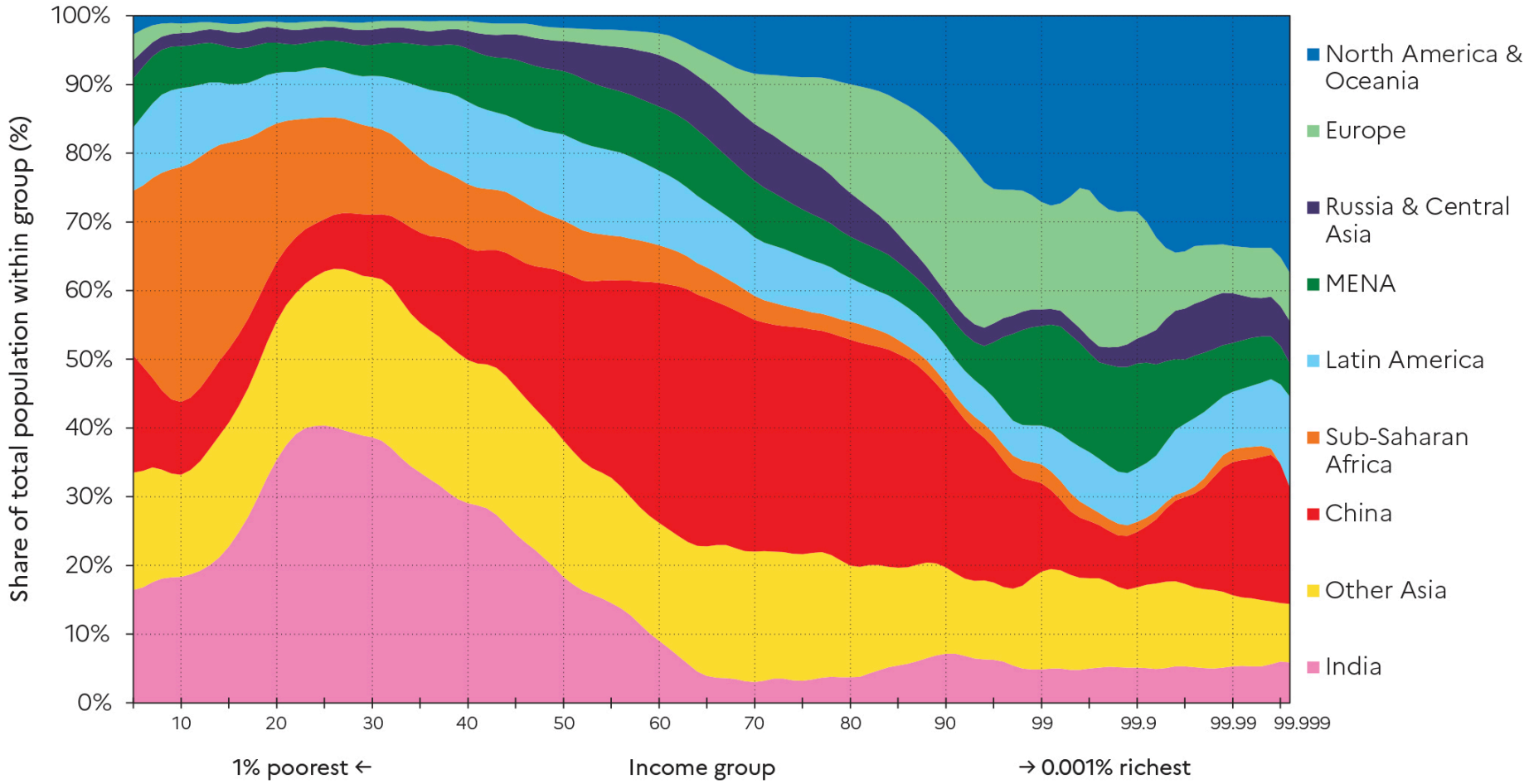
- Useful to compare and pool across countries, but can be misleading in interpretation
- Income/wealth/etc. comes out of various stochastic processes, government interventions, and individual choices
- Hypotheticals can be helpful to think things through
  - → If we made Canadians 5x more productive proportionality, preserving Canada's gini how would it change these global measures?
  - → If we magically made India have the same income distribution as the US, how much would that change the global Gini?

Figure 1 ***Global income and wealth inequality, 2021***



Interpretation: *The global bottom 50% captures 8.5% of total income measured at Purchasing Power Parity (PPP). The global bottom 50% owns 2% of wealth (at Purchasing Power Parity). The global top 10% owns 76% of total Household wealth and captures 52% of total income in 2021. Note that top wealth holders are not necessarily top income holders. Incomes are measured after the operation of pension and unemployment systems and before taxes and transfers.* ***Sources and series:*** *wir2022.wid.world/methodology.*

Figure 1.8 ***Geographic Breakdown of global income groups in 2021***



Interpretation: *The graph shows the geographical breakdown of global income groups. In 2021, 18% of the population of the world's top 0.001% income group were residents of China. Income measured after pension and unemployment benefits are received by individuals, and before income and wealth taxes.* ***Sources and series:*** *wir2022.wid.world/methodology.*

Wealth Dynamics

Wealth Dynamics

- Theory tells us that Kesten processes will lead to power-law tails and hence typically high degrees of inequality
- Income inequality itself can also be skewed
  - → It is sometimes a power-law but unlikely to be due to Kesten-style dynamics because human capital doesn't seem multiplicative
  - → However, complementarities in production can take small differences in talent and amplify them
  - → e.g., see **Why has CEO Pay Increased So Much by Gabaix and Landier**
- Here we will explore additive income + Kesten-style dynamics of wealth

Key Components of Wealth Dynamics

- Income will follow simple auto-regressive style
- Returns on wealth will have an IID component and be multiplicative
- Savings will be a a portion of wealth, as in previous models
  - → Assume they only save if the wealth is above a certain level
  - → This is a simple way to model that many people do not save
- Our prediction is that this distortion at the bottom of the distribution leads to power-law tails

Savings and Wealth Process

- w_t is wealth
- Stochastic variables, which may have an underlying state
  - → y_t is income, which will be stochastic
  - → R_{t+1} is the gross return on wealth, which will be stochastic
- The total income saved is an exogenous $s(w_t)$. Consumption implied

$$w_{t+1} = R_{t+1} s(w_t) + y_{t+1}$$

- Wealth net of consumption, $s(w_t)$, is a simple threshold

$$s(w_t) = \begin{cases} s_0 w_t & \text{if } w_t > \hat{w} \\ 0 & \text{otherwise} \end{cases}$$

Income and Returns Processes

- We will introduce a correlation between income and returns based on an underlying state

$$z_{t+1} = az_t + b + \sigma_z \epsilon_{t+1}$$

- Given this latent state the returns have IID shocks

$$R_t := 1 + r_t = c_r \exp(z_t) + \exp(\mu_r + \sigma_r \xi_t)$$

- And income has a similar IID component

$$y_t = c_y \exp(z_t) + \exp(\mu_y + \sigma_y \zeta_t)$$

- Where $\epsilon_t, \xi_t, \zeta_t$ are IID and standard normal

Creation of Model Parameters

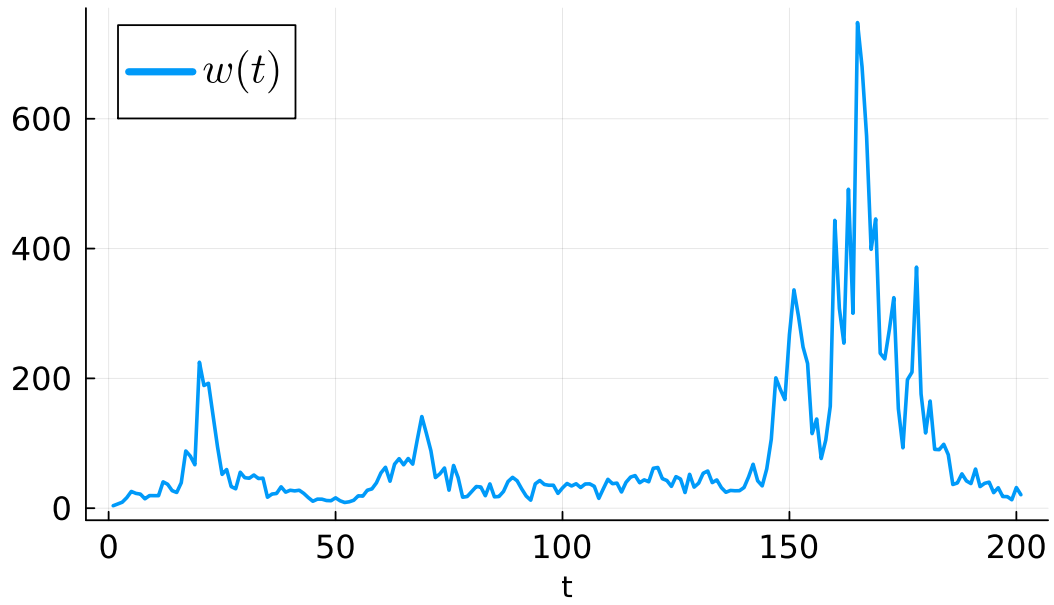
```
function wealth_dynamics_model(; # all named arguments
                               w_hat = 1.0, s_0 = 0.75, # savings
                               c_y = 1.0, mu_y = 1.0, sigma_y = 0.2, # labor income
                               c_r = 0.05, mu_r = 0.1, sigma_r = 0.5, # rate of return
                               a = 0.5, b = 0.0, sigma_z = 0.1)
    z_mean = b / (1 - a)
    z_var = sigma_z^2 / (1 - a^2)
    exp_z_mean = exp(z_mean + z_var / 2)
    R_mean = c_r * exp_z_mean + exp(mu_r + sigma_r^2 / 2)
    y_mean = c_y * exp_z_mean + exp(mu_y + sigma_y^2 / 2)
    alpha = R_mean * s_0
    z_stationary_dist = Normal(z_mean, sqrt(z_var))
    @assert alpha <= 1 # check stability condition that wealth does not diverge
    return (; w_hat, s_0, c_y, mu_y, sigma_y, c_r, mu_r, sigma_r, a, b, sigma_z,
            z_mean, z_var, z_stationary_dist, exp_z_mean, R_mean, y_mean, alpha)
end
```

Simulation of an Agent

```
function simulate_wealth_dynamics(w_0, z_0, T, params)
    (; w_hat, s_0, c_y, mu_y, sigma_y, c_r, mu_r, sigma_r, a, b, sigma_z) = params # unpack
    w = zeros(T + 1)
    z = zeros(T + 1)
    w[1] = w_0
    z[1] = z_0
    for t in 2:(T + 1)
        z[t] = a * z[t - 1] + b + sigma_z * randn()
        y = c_y * exp(z[t]) + exp(mu_y + sigma_y * randn())
        w[t] = y # income goes to next periods wealth
        if w[t - 1] >= w_hat # if above minimum wealth level, add savings
            R = c_r * exp(z[t]) + exp(mu_r + sigma_r * randn())
            w[t] += R * s_0 * w[t - 1]
        end
    end
    return w, z
end
```

Example Simulation

```
p = wealth_dynamics_model() # defaults
y_0 = p.y_mean
z_0 = rand(p.z_stationary_dist)
T = 200
w, z = simulate_wealth_dynamics(y_0, z_0, T, p)
plot(w, caption = "Wealth simulation",
        xlabel = "t", label = L"w(t)",
        size = (600, 400))
```



Ternary Operator

```
function f1(x)
    val = 2.0
    if x >= 0.0
        val += x
    else
        val -= x
    end
    return val
end
function f2(x)
    temp = (x >= 0.0) ? x : -x
    return 2.0 + temp
end
f3(x) = 2.0 + ((x >= 0.0) ? x : -x)
@show f1(0.8), f2(0.8), f3(0.8)
@show f1(1.8), f2(1.8), f3(1.8);
```

```
(f1(0.8), f2(0.8), f3(0.8)) = (2.8, 2.8, 2.8)
(f1(1.8), f2(1.8), f3(1.8)) = (3.8, 3.8, 3.8)
```

Simulate Panel with Ensemble

```
function simulate_panel(N, T, p)
    (; w_hat, s_0, c_y, mu_y, sigma_y, c_r, mu_r, sigma_r, a, b, sigma_z) = p
    w = p.y_mean * ones(N) # start at the mean of y
    z = rand(p.z_stationary_dist, N)
    zp = similar(z)
    wp = similar(w)
    R = similar(w)
    for t in 1:T
        z_shock = randn(N)
        R_shock = randn(N)
        w_shock = randn(N)
        @inbounds for i in 1:N
            zp[i] = a * z[i] + b + sigma_z * z_shock[i]
            R[i] = (w[i] >= w_hat) ? c_r * exp(zp[i]) + exp(mu_r + sigma_r * R_shock[i]) : 0.0
            wp[i] = c_y * exp(zp[i]) + exp(mu_y + sigma_y * w_shock[i]) + R[i] * s_0 * w[i]
        end
        w .= wp
        z .= zp
    end
```

Gini and median Wealth

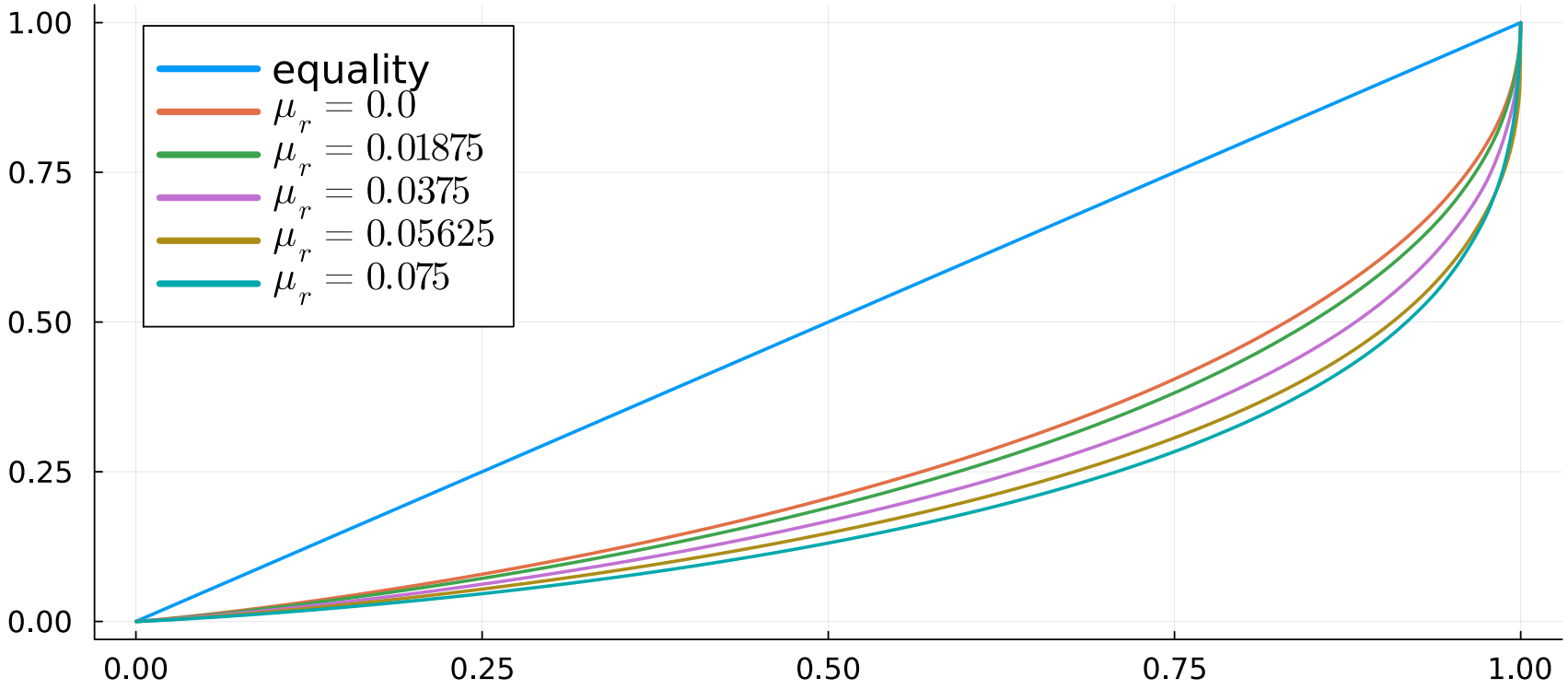
```
1 p = wealth_dynamics_model()
2 N = 10_000
3 T = 500
4 res = simulate_panel(N, T, p)
5 @show median(res.w)
6 @show res.gini;
```

```
median(res.w) = 38.865197453275826
res.gini = 0.7411006654233162
```

Lorenz Curves and Returns on Wealth

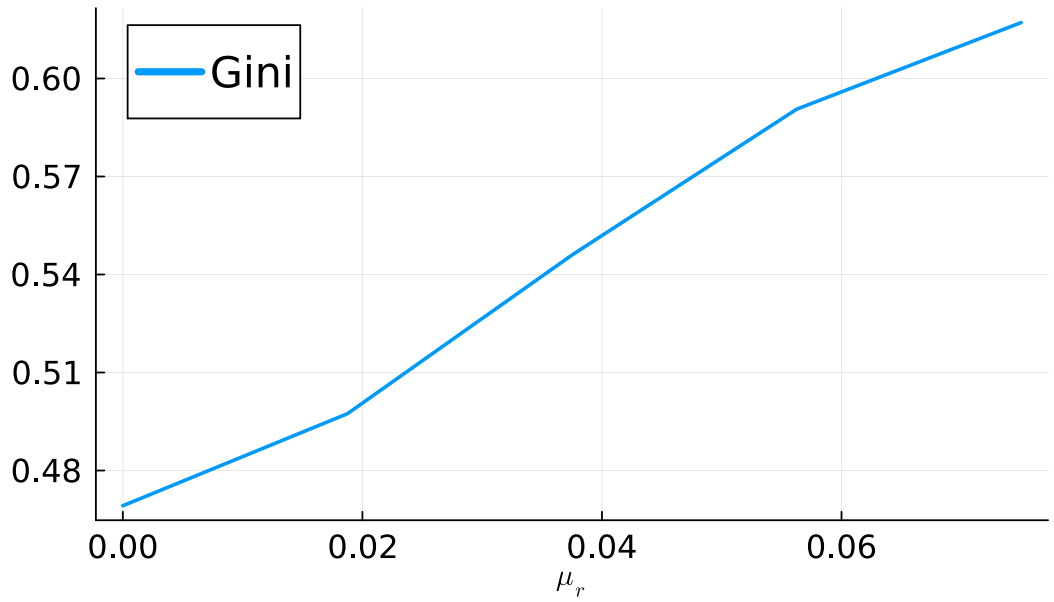
```
mu_r_vals = range(0.0, 0.075, 5)
results = map(mu_r -> simulate_panel(N, T, wealth_dynamics_model(; mu_r)),
              mu_r_vals);
plt = plot(results[1].F, results[1].F, label = "equality", legend = :topleft)
[plot!(plt, res.F, res.L, label = L"\mu_r = %$mu_r")
 for (mu_r, res) in zip(mu_r_vals, results)]
plt
```

Lorenz Curves and Returns on Wealth



Gini Coefficients

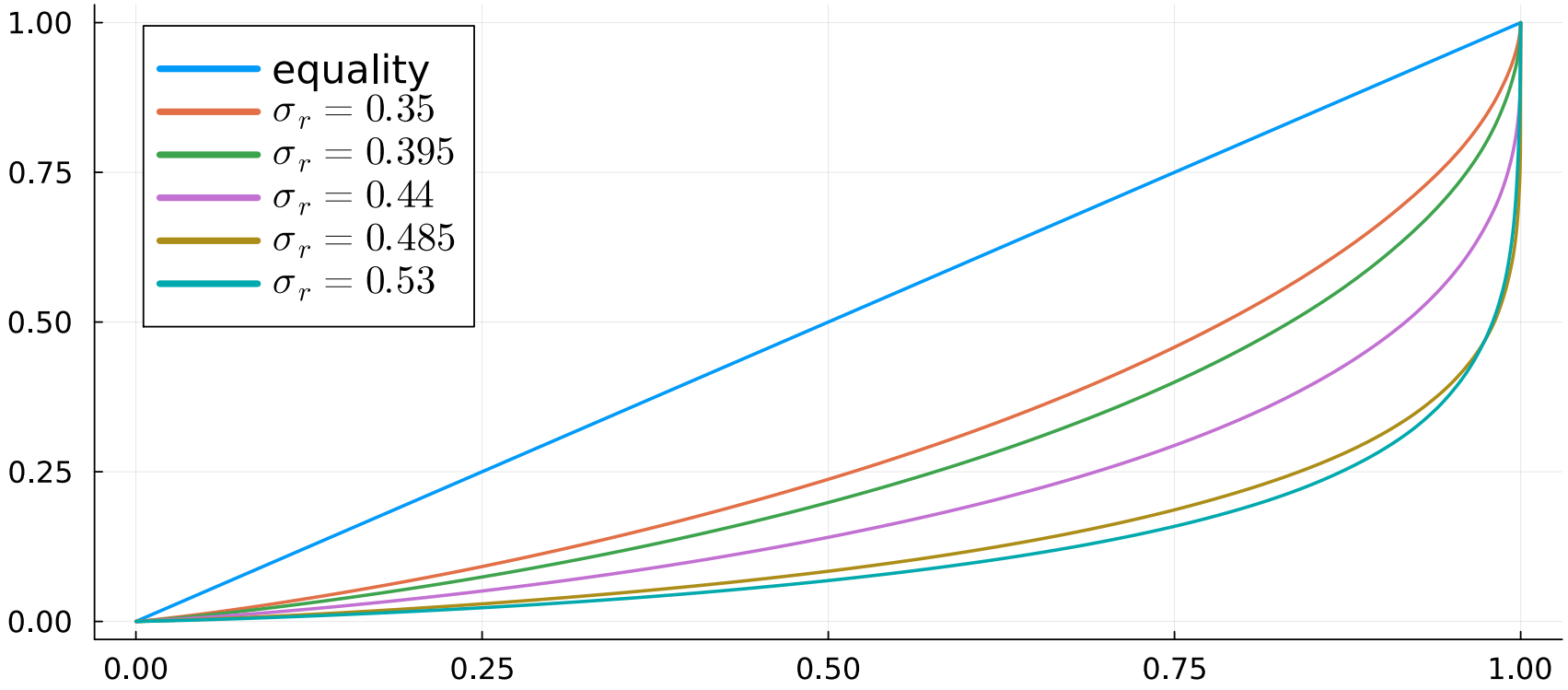
```
ginis = [res.gini for res in results]
plot(mu_r_vals, ginis;
    label = "Gini", xlabel = L"\mu_r",
    size = (600, 400))
```



Lorenz Curves and Volatility

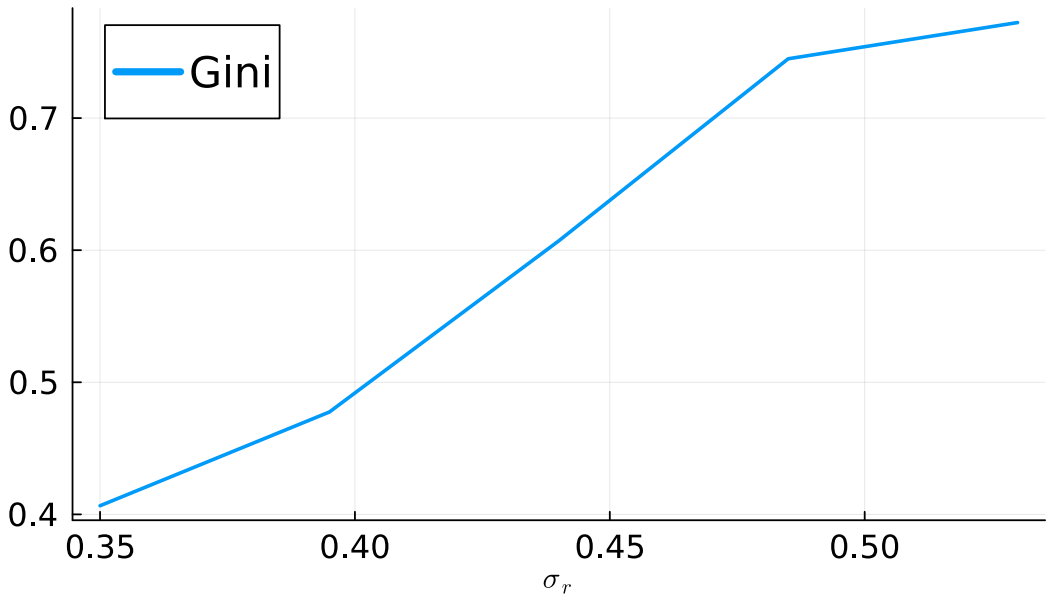
```
sigma_r_vals = range(0.35, 0.53, 5)
results = map(sigma_r -> simulate_panel(N, T, wealth_dynamics_model(; sigma_r)),
              sigma_r_vals);
plt = plot(results[1].F, results[1].F, label = "equality", legend = :topleft)
[plot!(plt, res.F, res.L, label = L"\sigma_r = %$sigma_r")
 for (sigma_r, res) in zip(sigma_r_vals, results)]
plt
```

Lorenz Curves and Volatility



Gini Coefficients

```julia
ginis = [res.gini for res in results]
plot(sigma_r_vals, ginis;
    label = "Gini", xlabel = L"\sigma_r",
    size = (600, 400))
```



(Optional) Benchmarking Examples

In-place Functions, Preallocation, and Performance

- One performance advantage of Julia is its ability to manage allocations and perform in-place operations.
- Don’t prematurely optimize your code - but in cases where the datastructures are large and the code is of equivalent complexity, don’t be afraid to use in-place operations.
- The convention in Julia is to use `!` to denote a function which mutates its arguments and to put any arguments that will be modified first.

Use Benchmarking when Performance Matters

- The **BenchmarkTools.jl** is a good package for benchmarking and performance
- Suggestions (after the "never prematurely optimize" rule)
  - → Always put the code in a function, especially loops
  - → Use the `$` to interpolate the variables into the function, avoiding global scope and global variables
  - → Only benchmark when you need speed, otherwise go for clarity
  - → Careful not to change types of variables (e.g. start as an integer, change to float)

Example of Benchmarking

```
f(x) = x.^2 .+ 5
x = rand(1_000)
@btime f($x)
# For more details
@benchmark f($x)
```

Example of Benchmarking

```
  474.510 ns (3 allocations: 7.88 KiB)
BenchmarkTools.Trial: 10000 samples with 280 evaluations.
 Range (min … max):  501.936 ns … 49.274 μs  ┊ GC (min … max):  0.00% … 18.56%
 Time  (median):     587.811 ns              ┊ GC (median):     0.00%
 Time  (mean ± σ):   903.527 ns ±  1.499 μs  ┊ GC (mean ± σ):  23.17% ± 15.47%
```

```
 502 ns        Histogram: log(frequency) by time      6.07 μs <
Memory estimate: 7.88 KiB, allocs estimate: 3.
```

In-place Lorenz

```
function lorenz!(L, v)
    cumsum!(L, v)
    L ./= L[end]
    F = (1:length(v))/length(v)
    # inplace can still return
    return F, L
end
n = 1_000_000
v = sort(rand(n) .^ (-1 / 2))
@btime lorenz($v)
L = similar(v)
@btime lorenz!($L, $v);
```

```
1.397 ms (6 allocations: 15.26 MiB)
1.352 ms (0 allocations: 0 bytes)
```

Performance Advantage?

- Depends on the system
- Memory allocations will be smaller regardless
- In-place operations can be faster, but not always, especially for small data.

Worth the Trouble? Another Inplace Example

```
1  mul_test!(Z, X, Y) = mul!(Z, X, Y)
2  mul_test(X, Y) = X * Y
3  n = 500
4  X = randn(n, n)
5  Y = randn(n, n)
6  Z = similar(Y)
7  # out-of-place multiplication
8  @btime mul_test($X, $Y)
9   # in-place multiplication
10 @btime mul_test!($Z, $X, $Y);
```

```
5.402 ms (3 allocations: 1.91 MiB)
5.396 ms (0 allocations: 0 bytes)
```