Rational Expectations and Markov Perfect Equilibrium

Undergraduate Computational Macro

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Overview

Motivation

- We have been studying many problems where agents use their expectations of the future to make decisions today
- But what is the feedback between those decisions, expectations, and the actual outcomes?
- This lecture introduces the concept of **rational expectations equilibrium** and **Markov perfect equilibrium** to study this feedback
- To illustrate it, we describe a linear quadratic version of a famous and important model due to Lucas and Prescott 1971

Materials

- Adapted from QuantEcon lectures coauthored with John Stachurski and Thomas J. Sargent
  - → Rational Expectations Equilibrium
  - → Markov Perfect Equilibrium

```julia
using LinearAlgebra, Statistics
using Distributions, LaTeXStrings, QuantEcon
using Plots.PlotMeasures, NLsolve, Roots, Random, Plots
default(;legendfontsize=16, linewidth=2, tickfontsize=12,
         bottom_margin=15mm)
```

Perceived and Actual Laws of Motion

Recall: Rational Expectations

- In previous lectures we discussed models for how agents' form expectations
- Our key approach was to use the mathematical expectation to formalize how agents' use expectations of the future to make decisions today
  - → e.g. , $\mathbb{E}_t[X_{t+j}]$ where X_{t+j} is a random variable forecast by agents given (precisely defined) information sets at time t
- It was often convenient to write down models with Markov random variables so that X_{t+1} only depended on X_t and not X_{t-1}, etc.
- If we assumed that agents' were using the best possible forecasts given the information they had, we called this **rational expectations**
  - → Key feature: **rational expectations** forecasts are not systematically biased

Microfoundations, Decisions, and Expectations

- We took those tools to take agents' forecasts of some stochastic process and solve for their decisions
- A core insight of modern macroeconomics is that in order to evaluate changes in policy you need to have some self-consistent model of behavior (what we call **microfoundations**)
- These give us a tool to understand **counterfactuals** of how an agent would respond to changes in the environment (e.g., endogenous savings in the Permanent Income Model)
- If we take as given the stochastic processes that agents use to form expectations (e.g., paths of prices such as interest rates) we can evaluate how changes to that environment will change the decisions of agents

Decisions and Outcomes

- But what if the stochastic process is a consequence of those decisions?
- e.g. there are agents' $i = 1, \ldots I$ each with a state X_t^i and a decision that affects each individuals payoffs, then agent i must forecast the entire distribution of X_{t+1}^j for $j \neq i$ to make their decision
  - → This, in turn, requires them to know the decision rules of those agents
- Sometimes this can be simplified since "payoff relevant" (i.e., things that enter individual decisions) are summarized by prices like interest rates, future wages, etc.
- Keep in mind that the true evolution of X_{t+1}^j will depend on the decisions of all agents $j = 1, \ldots I$, so if we consider policy changes that affect decisions, it would change the evolution of X_{t+1}^j

Example: Capital Accumulation

- For example, consider savings decisions in a **growth model**
- As we saw, the rate of savings by consumers will determine the rate of capital accumulation
- But in a competitive market, the return to capital will depend on the amount of capital in the economy
- From the consumers' perspective, they will make decisions based on the returns to capital, which are a price like an interest rate
- In order to make optimal savings decisions they then need to forecast the future capital stock (or interest rates), which in turn are a consequence of those savings

Example: Education and Wages

- Or another example, consider how much education (or other human capital) a consumer should acquire
- When making human capital decisions, students try to predict future wages for different levels of education
- But those wages depend on the supply of workers with a given level of education
  - → So in some ways they need to forecast the decisions of other students to make their own decisions - which could be used to forecast the wages
- To consider policy counterfactuals large enough to change to education distribution (and wages) we need students to be able to forecast how other students will respond to those changes

Example: Asset Pricing

- Our classic **consumption-based asset pricing** results are similar
- Part of the decision on the price of an asset today is the expected future price of that asset
  → But that price was determined by the decisions of all other agents!
- We considered the case where the pricing was self-consistent (easy in that case because we assumed a large number of identical agents)
  → We solved a version with endowments (i.e. stochastic process c_t in fixed supply) but if production or other decisions entered, then agents' would have needed to predict them to make their decisions

Example: Firm Investment

- A classic example comes from Lucas and Prescott (1971) where firms make investment decisions
- To determine the appropriate scale of a firm, firms need to forecast profits
- But prices and profits depend on the scale of all firms in the economy
  - → The market clearing price is decreasing in the total output of all firms
  - → Which in turn depends on the investment decisions
- So firms' need to forecast the decisions of all other firms (or, at least, forecast the prices) to make their own investment decision

Expectations and Outcomes

- If you consider counterfactuals which are unlikely to change aggregates (and hence prices) much, then this feedback isn't a problem
  - → i.e., **partial equilibrium** analysis which is fine for many questions
- But if not, then this feedback is especially challenging because it relies on **expectations** of the future
- This is at the heart of why macroeconomics is theoretically and computationally challenging
  - → It also forces you to take a stand on models of expectations
- Without assumptions, you may end up with internally inconsistencies (e.g., agents' forecasts of prices are based on an evolution of the economy that is inconsistent with those decisions)

Equilibrium Concepts

Rational Expectations Equilibrium

- A classic benchmark which preserves this self-consistency is the **rational expectations equilibrium** (REE)
  1. Agents' make decisions today using **rational expectations** to forecast the future - and taking as given the stochastic process of the evolution of the aggregate state/etc.
  2. Given those decisions, solve for the actual evolution of the aggregate state
  3. The equilibria is that the perceived and actual laws of motion are the same (i.e., *fixed-point between the perceived and actual laws of motion*)
- As with rational expectations, agents do not know the future exactly, but forecast given the information they have and to have no systematic biases

Markov Perfect Equilibrium

- This is a special case of another equilibrium concept called **Markov Perfect Equilibrium** (MPE) or Markov Perfect Nash Equilibrium
- In that, we make the same basic assumptions but consider strategic interactions between agents
  - → e.g., in a Nash equilibrium, if I change my strategy today it will change the aggregate state, which will change the strategies of other agents
  - → e.g., if I change my output, I am big enough to affect the aggregates
- With a smaller number of agents (e.g., 3 firms completing) MPE is more appropriate - but in the limit they converge
- Intuitively: as soon as the actions of each agent end up having a very small affect on payoff relevant functions of the the aggregate state (e.g., prices) then the MPE and REE are often the same

Aggregates and Representative Agents

- REE are typically used with **price-taking** agents (i.e., competitive equilibrium)
- With many competitive equilibria, the distribution is not important, and the aggregate state is sufficient to calculate "payoffs"
- e.g., a firm may only need to know the total amount of output of competitive firms, not the distribution of output
- If we do not need to solve for the evolution of the distribution, then we may be able to assume conditions such that a **representative agent** can be used to solve the model. Aggregation result (not assumption)
  - → The easiest examples are often with **homothetic** and identical production processes and preferences

The “Big K, little k” trick

- The self-consistency between the perceived and actual laws of motion is easiest to see work with when all firms are identical and price-taking
- Classic example: Lucas and Prescott (1971) model of firm investment
  - → The perceived law of motion is of the aggregate K, capital
  - → The individual firm chooses investment k to maximize profits
  - → The REE is when the perceived law of motion is the same as the actual law of motion (i.e. $K = Nk$ for N identical firms)
  - → If all we care about are dynamics of the aggregate K then we might as well just use a single representative firm
- Pervasive technique in macro: often called the “Big K, little k” trick

Simple Example

Static Example

- To demonstrate the approach, lets first consider a simple static example
- There is an undifferentiated good produced by $n = 1, \ldots N$ firms sold on a competitive market for p
- N firms each face cost to produce with y units of output

$$c(y) = c_1 y + \frac{N}{2} c_2 y^2$$

  → (Scaling by N is a hack to make things closer to homothetic)
- If firms are identical, then define the aggregate output $Y \equiv \sum_{n=1}^{N} y_n = Ny$
  → Where the last equality is only true if all firms are identical

Demand and Price

- Since the good is undifferentiated, and if we assume that prices are determined by the aggregate supply and demand, we can summarize demand as an inverse-demand function

$$p(Y) \equiv a_0 - a_1 \sum_{n=1}^{N} y_n = a_0 - a_1 Y$$

  → Where a_0 and a_1 are parameters

  → Note that this is a function of the **aggregate** output

Profit Maximization for REE

- Taking prices $p(Y)$ and the perceived output of the other firms (Y) as given the firm solves

$$\max_{y} \{p(Y)y - c(y)\}$$

  → Crucially, the Y and the y are different here! The Y is their perceived output level in the industry, and y is the firms' chosen output level

  → Here we are thinking that the firm is sufficiently small, or non-strategic, such that it doesn't consider its own impact on the price

Take the FOC First, Apply REE Second

- Take the problem with the cost and prices

$$\max_{y}\left\{(a_0 - a_1 Y)y - c_1 y - \frac{N}{2}c_2 y^2\right\}$$

- Take the FOC with respect to the y. Don't touch the Y! They take it as given

$$a_0 - a_1 Y - c_1 - c_2 N y = 0$$

- Finally, to impose the REE we have that $Y = Ny$ so that

$$a_0 - a_1 Y - c_1 - c_2 Y = 0$$

→ Or that $Y = \frac{a_0 - c_1}{a_1 + c_2}$. Why not just use $N = 1$?

Nash Equilibrium-Style Solutions

- The MPE/Nash equivalent in this case would be if the firm considers that its own output will affect the price
- In particular, firm n forecasts y_j for all $j \neq n$ and then solves

$$\max_{y_n} \left\{ p\left(\sum_{j \neq n} y_j + y_n \right) y_n - c(y_n) \right\}$$

  → Which requires considering the impact of $\partial_{y_n} p(Y)$ in their decision

  → Then an symmetric MPE is one in which the y_n are all the same and fulfill that equation

- With large N, $\partial_{y_n} p(Y) \approx 0$ and an REE is a good approximation

Dynamic Firm Investment Model

Competitive Equilibria with Adjustment Costs

- Maintain capital = output for now. Just trying to make the
  - → y_t^n for $n = 1, \ldots N$ firms
  - → $Y_t = \sum_{n=1}^N y_t^n$, aggregate output
- Homogenous good sold at price $p(Y) = a_0 - a_1 Y$
- Firms are profit maximizing **price takers** discounting at rate β

Adjustment Costs

- While they have no uncertainty, it is costly to adjust their output
- The cost of adjusting output is quadratic in the change in output

$$c(y_{t+1} - y_t) = \frac{1}{2}\gamma(y_{t+1} - y_t)^2$$

  → Where $\gamma > 0$ is a parameter

- Since they own the capital to produce, we will ignore other marginal production costs. Normalizing

Forecasting the Aggregates

- The challenge faced by the firm is thatp_t depend on aggregate output Y_t, will drop and perhaps even go negative
- So if they over produce and choose too high of a y_t then may end up with lower or negative profits
- The REE challenge is that all other firms are making that decision, so Y_t is a function of all the y_t's
- We could have them forecast p_t directly, but note that $p_t = p(Y_t)$ so we can have them forecast a law of motion for Y_t instead

Firm Beliefs

- The agents have a deterministic **perceived law of motion** for Y_t
  - → More generally, we would need a stochastic process for the perceived law of motion of Y_t.
- Given the known initial condition Y_0, define this as

$$Y_{t+1} = H(Y_t)$$

- This belief $H(\cdot)$ is an equilibrium object we will solve for
  - → The next step is to take it as given, and solve the firm's problem

Sequential Firm Problem (Given Beliefs)

- Take $H(\cdot)$ **as given** and generate Y_t from $H(\cdot)$ and Y_0
- Firm period profits net adjustment costs are $p(Y_t)y_t - c(y_{t+1}, y_t)$
- The firm takes prices, y_0, Y_0 and given, and solves

$$\max_{\{y_{t+1}\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \left(p(Y_t)y_t - c(y_{t+1}, y_t) \right) \right\}$$

→ Turns out we can write solution as a $y_{t+1} = h(y_t, Y_t)$ in this case.

Firm Problem (Given Beliefs)

- Jumping to a Bellman Equation with value function: $v(y, Y)$
- The state: y for the firm, and Y for the aggregate. Take $H(\cdot)$ **as given**

$$
\begin{aligned}
v(y, Y) &= \max_{y'} \left\{ p(Y)y - c(y', y) + \beta v(y', H(Y)) \right\} \\
&= \max_{y'} \left\{ a_0 y - a_1 y Y - \frac{\gamma(y' - y)^2}{2} + \beta v(y', H(Y)) \right\}
\end{aligned}
$$

→ Define the solution to the optimization problem as the $\arg\max$, and denote $y' = h(y, Y)$

- Note that $v(y, Y)$ and $h(y, Y)$ would depend on the $H(Y)$

(Optional) First-Order Characterization of h

- Take the FONC, then

$$-\gamma(y' - y) + \beta v_y(y', H(Y)) = 0$$

  → As in **asset pricing** we need to use the envelope theorem, following a result of **Benveniste-Scheinkman (1979)**

  → Gives that $v_y(y, Y) = a_0 - a_1 Y + \gamma(y' - y)$

- Substituting this equation gives

$$-\gamma(y_{t+1} - y_t) + \beta[a_0 - a_1 Y_{t+1} + \gamma(y_{t+2} - y_{t+1})] = 0$$

  → with initial conditions for (y_0, Y_0)

  → the terminal **transversality condition** $\lim_{t\to\infty} \beta^t y_t v_y(y_t, Y_t) = 0$

Imposing the REE Equilibrium

- Given a solution for the $y_{t+1} = h(y_t, Y_t)$, we can then impose the REE equilibrium

$$
\begin{aligned}
Y_{t+1} &= H(Y_t) \\
&= Nh(Y_t/N, Y_t)
\end{aligned}
$$

- In this case, can show that the Y_t evolution will be independent of N
- Might as well set $N = 1$ then set $Y_t = y_t$ to solve for the equilibrium
  - → i.e. $h(Y_t, Y_t) = H(Y_t)$, or the actual LOM = perceived LOM
  - → The "Big K, little k" trick. But can't do $Y_t = y_t$ until **after** FOCs

Computing the REE

Solution Methods

- This fixed point problem connecting the perceived and actual laws of motion is computationally challenging
  - → Internally we can fix a LOM and then solve the agent's problem, but then we need to find the new LOM given those decisions
  - → We can literally set it up as a fixed-point problem, but it isn't a **contraction mapping** so naive iteration won't always work.
- In some cases we can find a **planning problem** with the same solution
- Here we will discuss the general approach without going through the computational challenges and solutions

Outline of Algorithm

1. Choose a functional form for the LOM. In this case, we will guess that

$$Y_{t+1} = H(Y_t) = \kappa_0 + \kappa_1 Y_t$$

2. Then, for some κ_0 and κ_1 guess, we solve the firm's problem taking the $H(Y_t)$ as given
   - This gives us a $y_{t+1} = h(y_t, Y_t) = h_0 + h_1 y_t + h_2 Y_t$ in this case
3. Use the $N = 1$ example, and set $Y_t = y_t$ and find the implied κ_0 and κ_1

$$\begin{aligned} Y_{t+1} &= H(Y_t) = \kappa_0 + \kappa_1 Y_t \\ N y_{t+1} &= N h(y_t, Y_t) = N h_0 + N h_1 y_t + N h_2 Y_t \end{aligned}$$

4. Applying the simple $N = 1$ case we have $y_t = Y_t$ which means that

$$\kappa_0 + \kappa_1 Y_t = h_0 + h_1 Y_t + h_2 Y_t$$

5. Iterate until $\kappa_0 \approx h_0$ and $\kappa_1 \approx h_1 + h_2$

Linear Quadratic Optimal Control

- This problem fits into the **Linear Quadratic (LQ) optimal control** framework

$$\max_{\{u_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(x_t^\top R x_t + u_t^\top Q u_t \right)$$
$$\text{s.t. } x_{t+1} = A x_t + B u_t$$

  - → Where x_t is a state vector, u_t is a control vector, and R, Q are matrices for payofffs
  - → The $x_{t+1} = A x_t + B u_t$ is a LSS law of motion. Also can handle gaussian shocks
- Can show optimal solution is a $u_t = -F x_t$ for some F matrix

Fitting the LQ Law of Motion

- **If** the perceived law of motion is $Y_{t+1} = H(Y_t) = \kappa_0 + \kappa_1 Y_t$ this fits the LSS
- Let $x_t \equiv \begin{bmatrix} y_t & Y_t & 1 \end{bmatrix}^\top$
- Let $u_t \equiv y_{t+1} - y_t$, i.e. scalarchange in output

$$\underbrace{\begin{bmatrix} y_{t+1} \\ Y_{t+1} \\ 1 \end{bmatrix}}_{\equiv x_{t+1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \kappa_1 & \kappa_0 \\ 0 & 0 & 1 \end{bmatrix}}_{\equiv A} \underbrace{\begin{bmatrix} y_t \\ Y_t \\ 1 \end{bmatrix}}_{\equiv x_t} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\equiv B} u_t$$

Fitting the LQ Payoffs

- Payoffs in our model are

$$p(Y_t)y_t - c(y_{t+1}, y_t) = a_0 y_t - a_1 y_t Y_t - \frac{\gamma(y_{t+1} - y_t)^2}{2}$$

- Mapping to the LQ setup

$$= \underbrace{\begin{bmatrix} y_t & Y_t & 1 \end{bmatrix}}_{\equiv x_t^\top} \underbrace{\begin{bmatrix} 0 & a_1/2 & -a_0/2 \\ a_1/2 & 0 & 0 \\ -a_0/2 & 0 & 0 \end{bmatrix}}_{\equiv R} \underbrace{\begin{bmatrix} y_t \\ Y_t \\ 1 \end{bmatrix}}_{\equiv x_t} + u_t^\top \underbrace{[\gamma/2]}_{\equiv Q} u_t$$

Mapping Solution to h and H

- LQ optimal control gives an optimal $u_t = -Fx_t$
- For our u_t and x_t,

$$
\begin{aligned}
u_t = y_{t+1} - y_t &= -F_1 y_t - F_2 Y_t - F_3 \\
y_{t+1} &= (1 - F_1) y_t - F_2 Y_t - F_3
\end{aligned}
$$

- Let $y_t = Y_t$ to apply REE condition for $N = 1$

$$
Y_{t+1} = -F_3 + (1 - F_1 - F_2) Y_t \equiv \hat{\kappa}_0 + \hat{\kappa}_1 Y_t
$$

- Compare to the assumed H LOM to solve

$$
Y_{t+1} = H(Y_t) = \kappa_0 + \kappa_1 Y_t
$$

Implementation of Firm Problem (Given Beliefs)

```
function solve_firm_problem(kappa, mod)
  kappa_0, kappa_1 = kappa # beliefs for H(Y_t)
  (; a_0, a_1, gamma, beta) = mod
  A = [1 0 0
      0 kappa_1 kappa_0
      0 0 1]
  B = [1.0, 0.0, 0.0]
  R = [0 a_1/2 -a_0/2
      a_1/2 0 0
      -a_0/2 0 0]
  Q = gamma/2
  lq = QuantEcon.LQ(Q, R, A, B; bet = beta)  # Package solves for u_t = -F x_t
  P, F, d = stationary_values(lq)
  kappa_0_hat = -F[3]
  kappa_1_hat = 1 - F[1] - F[2]
  return [kappa_0_hat, kappa_1_hat] # implied h(Y_t, Y_t)
end
```

Rational Expectations Equilibria

- Key: find fixed point of $H(Y) = h(Y, Y)$
- Note that naive fixed-point iteration may not work

```julia
function solve_REE(mod; kappa_iv = [95.5, 0.95], Y_ss_iv = [1500.0])
  sol = fixedpoint(kappa -> solve_firm_problem(kappa, mod), kappa_iv)
  kappa_0, kappa_1 = sol.zero
  H(Y) = [kappa_0 + kappa_1 * Y[1]] # vectorized
  Y_ss = fixedpoint(Y -> H(Y), Y_ss_iv).zero # steady state of H(Y)
  return (;kappa_0, kappa_1, H, Y_ss)
end
```

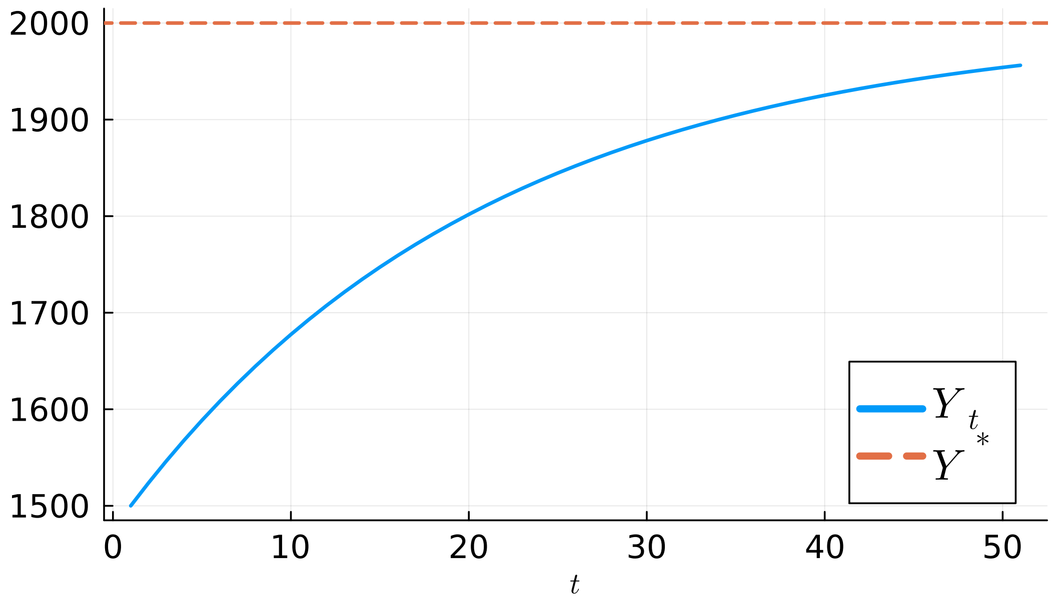
Finding the Fixed Point of H = h

```
firm_dynamics_model(;a_0 = 100, a_1 = 0.05, beta = 0.95, gamma = 10.0) = (;a_0, a_1, beta, gamma)
kappa_iv = [95.5, 0.95] # H() belief initial condition
mod = firm_dynamics_model() # default values
kappa_hat = solve_firm_problem(kappa_iv, mod)
(;kappa_0, kappa_1, H, Y_ss) = solve_REE(mod; kappa_iv)
@show solve_firm_problem([kappa_0, kappa_1], mod) - [kappa_0, kappa_1];
```

```
solve_firm_problem([kappa_0, kappa_1], mod) - [kappa_0, kappa_1] = [1.7195134205394424e-12, 1.1102230246251565e-15]
```

REEE Dynamics

```julia
function iterate_map(f, x0, T)
    x = zeros(length(x0), T + 1)
    x[:, 1] = x0
    for t in 2:(T + 1)
        x[:, t] = f(x[:, t - 1])
    end
    return x
end
(;H, Y_ss) = solve_REE(mod)
Y_0 = [1500.0]
T = 50
Y_path = iterate_map(H, Y_0, T) # H
plot(Y_path'; label = L"Y_t", xlabel = L"t",
     size = (600, 400), legend = :bottomright)
hline!([Y_ss[1]]; label = L"Y^*",
       linestyle=:dash)
```



Counterfactual

```
(;H, Y_ss) = solve_REE(mod)
Y_0 = [1500.0]
T = 200
Y_path = iterate_map(H, Y_0, T)
plot(Y_path'; label = L"Y_t(\gamma = 10.0)",
     xlabel = L"t", size = (600, 400),
     legend = :bottomright)
mod = firm_dynamics_model(;gamma = 20.0)
(;H, Y_ss) = solve_REE(mod)
Y_path = iterate_map(H, Y_0, T)
plot!(Y_path'; label = L"Y_t(\gamma = 20.0)")
hline!([Y_ss[1]]; label = L"Y^*",
       linestyle=:dash)
```