Markov Chains with Applications to Unemployment and Asset Pricing

Undergraduate Computational Macro

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Overview

Motivation

- Here we will introduce Markov Chains as a Markovian stochastic process over a discrete number of states
  - → These are useful in their own right, but are also a powerful tool if you discretize a continuous-state stochastic process
- Using these, we will apply these to
  - → Introduce a simple model of unemployment and employment dynamics
  - → Risk-neutral asset pricing
- In a future lecture these for more advanced asset-pricing examples including option-pricing and to explore risk-aversion

Materials

- Adapted from QuantEcon lectures coauthored with John Stachurski and Thomas J. Sargent
  - → Finite Markov Chains
  - → A Lake Model of Employment and Unemployment

```julia
using LinearAlgebra, Statistics, Distributions
using Plots.PlotMeasures, Plots, QuantEcon, Random
using StatsPlots, LaTeXStrings, NLsolve
default(;legendfontsize=16, linewidth=2, tickfontsize=12,
         bottom_margin=15mm)
```

Markov Chains

Discrete States

- Consider a set of N possible states of the world
- **Markov chain**: a sequence of random variables $\{X_t\}$ on $\{x_1, \ldots, x_N\}$ with the Markov property

$$\mathbb{P}(X_{t+1} = x \,|\, X_t) = \mathbb{P}(X_{t+1} = x \,|\, X_t, X_{t-1}, \ldots)$$

- It will turn out that all Markov stochastic processes with a discrete number of states are Markov Chains and can be summarize by a **transition matrix**

See here for Continuous Time Markov Chains which replace the transition probabilities with transition rates

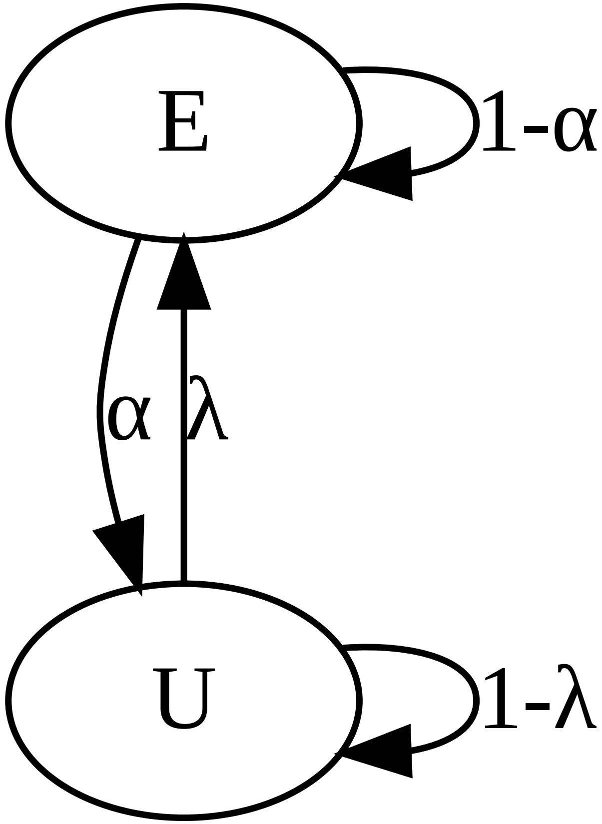
Transition Matrix

- Summarize into a $P \in \mathbb{R}^{N \times N}$ **transition matrix** where

$$P_{ij} \equiv \mathbb{P}(X_{t+1} = x_j \,|\, X_t = x_i), \qquad \text{for } i = 1, \ldots N, j = 1, \ldots N$$

- Each row is a probability distribution for the next state (j) conditional on the current one (i)
  - → Hence $P_{ij} \geq 0$ and $\sum_{j=1}^{N} P_{ij} = 1$ for all i
- The ordering of the matrix or states $x_1, \ldots x_N$ is arbitrary, but you need to be consistent!

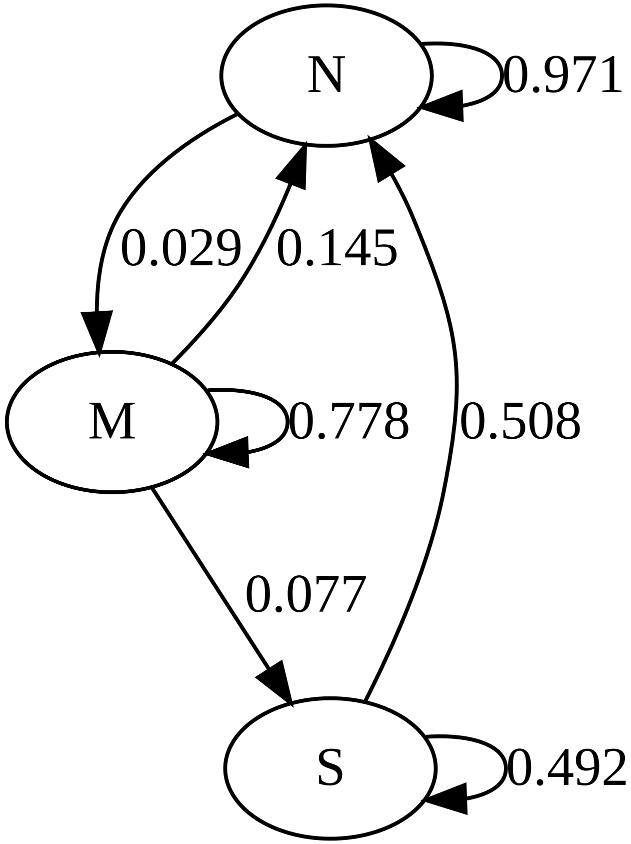
Example: Unemployed and Employed



- α: probability of moving from employed to unemployed
- λ: probability of moving from unemployed to employed
- $\mathbb{P}(X_{t+1} = U \,|\, X_t = E) = \alpha$, etc.
- Summarize as Transition Matrix

$$P \equiv \begin{bmatrix} 1-\alpha & \alpha \\ \lambda & 1-\lambda \end{bmatrix}$$

Example: Recessions Transitions



- States (ordered consistently):
  - → N: Normal Growth, M: Mild Recession, S: Severe Recession
- Transitions empirically estimated in Hamilton 2005

$$P \equiv \begin{bmatrix} 0.971 & 0.029 & 0 \\ 0.145 & 0.778 & 0.077 \\ 0 & 0.508 & 0.492 \end{bmatrix}$$

Discrete RVs

```julia
probs = [0.6, 0.4]
@show sum(probs) ≈ 1
d = Categorical(probs)
@show d
draws = rand(d, 4)
@show draws
# Assign associated with indices
G = [5, 20]
# access by index
@show G[draws];
```

```
sum(probs) ≈ 1 = true
d = Categorical{Float64, Vector{Float64}}
(support=Base.OneTo(2), p=[0.6, 0.4])
draws = [2, 1, 1, 1]
G[draws] = [20, 5, 5, 5]
```

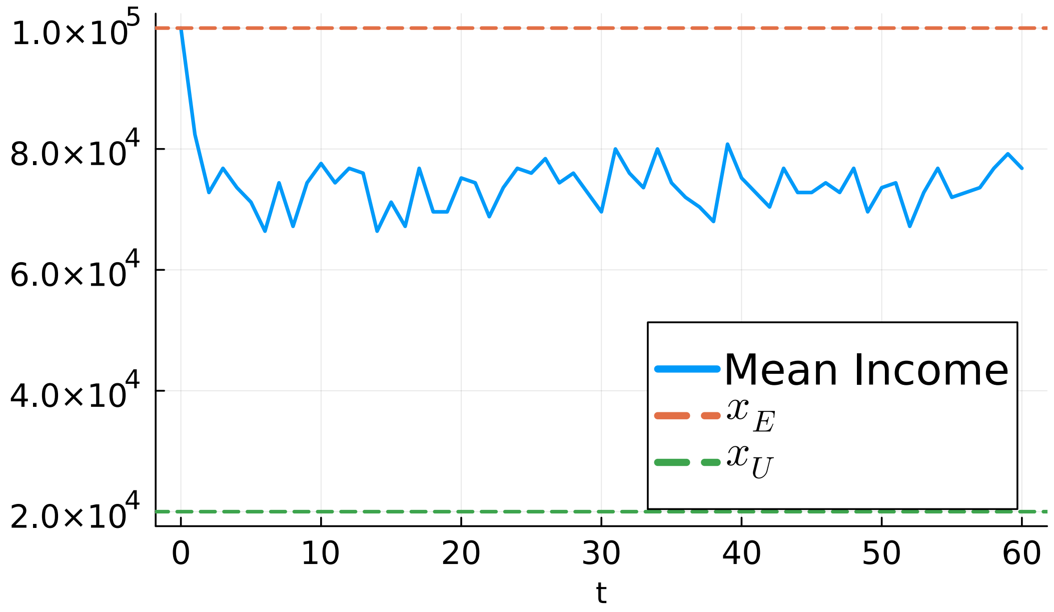
Simulating Markov Chains

```julia
function simulate_markov_chain(P, X_0, T)
    N = size(P, 1)
    num_chains = length(X_0)
    P_dist = [Categorical(P[i, :])
              for i in 1:N]
    X = zeros(Int, num_chains, T+1)
    X[:, 1] .= X_0
    for t in 1:T
        for n in 1:num_chains
            X[n, t+1] = rand(P_dist[X[n, t]])
        end
    end
    return X
end
```

- Create `Categorical` per row
- One chain for each `X_0`
- Simulate for each chain by:
  - → Save current index
  - → Use index to choose row
  - → Draw the new index according to that distribution

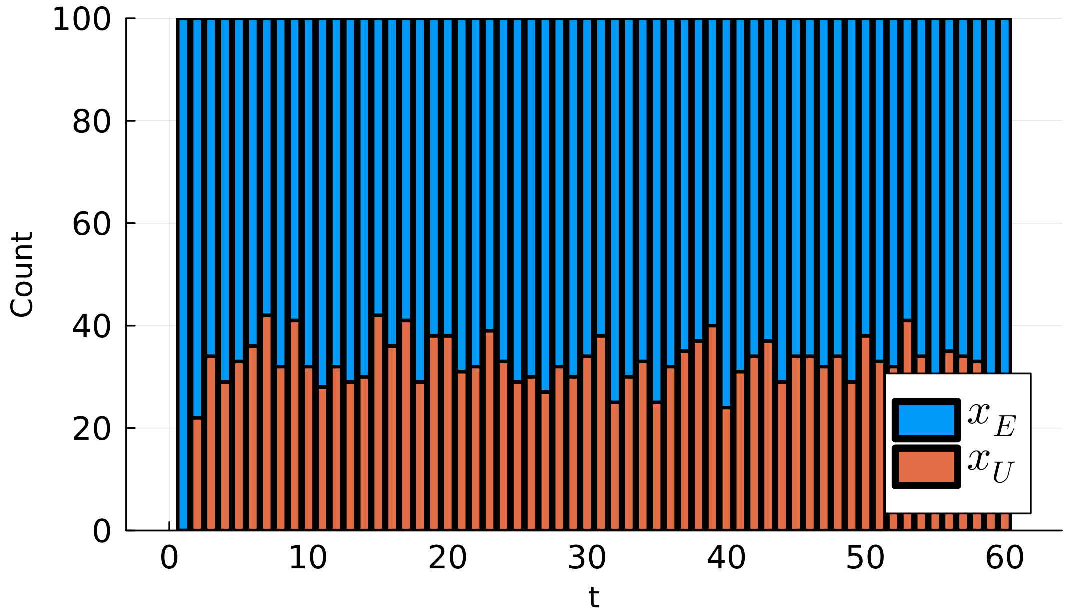
Simulating Unemployment and Employment

```
alpha, lambda = 0.3, 0.6
P = [1-alpha alpha; lambda 1-lambda]
G = [100000.00, 20000.00]
X_0 = ones(Int, 100)  # 100 people employed
T = 60
X = simulate_markov_chain(P, X_0, T)
X_values = G[X]  # just indexes by the X
X_mean = mean(X_values;dims=1)
plot(0:T, X_mean', xlabel="t",
     legend=:bottomright, label="Mean Income",
     size=(600, 400))
hline!([G[1]]; label=L"x_E", linestyle=:dash)
hline!([G[2]]; label=L"x_U", linestyle=:dash)
```



Distribution of Future Wages

```
unique_values = unique(X_values)
counts = [sum(X_values[:, t] .== val) for
          val in unique_values, t in 1:T]
# Create the stacked bar chart
groupedbar(1:T, counts';
           bar_position = :stack,
           xlabel="t", ylabel="Count",
           label = [L"x_E" L"x_U"],
           size=(600, 400))
```



Simulating with QuantEcon packages

```julia
alpha, lambda = 0.3, 0.6
P = [1-alpha alpha; lambda 1-lambda]
mc = MarkovChain(P)
T = 1000
init=1 # initial condition
# using QuantEcon.jl
X = simulate(mc, T;init)
prop_E = sum(X .== 1)/length(X)
println("Prop in E = $prop_E");
```

```
Prop in E = 0.69
```

Transitions and Expectations

Probability Mass Functions (PMF)

- Let the PMF of X_t be given by a row vector

$$\pi_t \equiv \begin{bmatrix} \mathbb{P}(X_t = x_1) & \dots & \mathbb{P}(X_t = x_N) \end{bmatrix}$$

  - → $\pi_{ti} \geq 0$ for all $i = 1, \dots N$ and $\sum_{i=1}^{N} \pi_{ti} = 1$
  - → Using π_t a row vector for convenience
- If the initial state is known at $t = 0$ then π_0 might be degenerate
  - → e.g., if $\mathbb{P}(X_0 = E) = 1$ then $\pi_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$

Conditional Forecasts

- Many macro questions involve: $\mathbb{P}(X_{t+j} = x_i | X_t = x_j)$ etc.
- The transition matrix makes it very easy to forecast the evolution of the distribution. Without proof, given π_t initial condition

$$[\mathbb{P}(X_{t+1} = x_1) \quad \ldots \quad \mathbb{P}(X_{t+1} = x_N)] \equiv \pi_{t+1} = \pi_t P$$

- Inductively: for the matrix power (i.e. $P \times P \times \ldots P$, not pointwise)

$$[\mathbb{P}(X_{t+j} = x_1) \quad \ldots \quad \mathbb{P}(X_{t+j} = x_N)] \equiv \pi_{t+j} = \pi_t P^j$$

Conditional Expectations

- Given the conditional probabilities, expectations are easy
- Now assign X_t as a random variable with values $x_1, \ldots x_N$ and pmf π_t
- Define $G \equiv \begin{bmatrix} x_1 & \ldots & x_N \end{bmatrix}$
- From definition of conditional expectations, where $X_t \sim \mu_t$

$$\mathbb{E}[X_{t+j} \,|\, X_t] = \sum_{i=1}^{N} x_i \pi_{t+j,i} = G \cdot (\pi_t P^j) = G(\pi_t P^j)^\top$$

- This works for **enormous** numbers of states N, as long as P is sparse (i.e., the number of elements of P is significant)

Example: Expected Income

- Define incomes in E and U states as
  - $\boldsymbol{G \equiv [100,000 \quad 20,000]}$
  - Maintain $\mathbb{P}(\boldsymbol{X_0 = E}) = \mathbf{1}$, or $\boldsymbol{\pi_0} = [1 \quad 0]$
- Expected income in 20 periods is then

$$\mathbb{E}[X_{20} \mid X_0 = x_E] = G \cdot (\pi_0 P^{20})$$

Reminder: PDV for Linear State Space Models

- If $x_{t+1} = Ax_t + Cw_{t+1}$ and $y_t = Gx_t$ then,

$$
\begin{aligned}
p(x_t) &= \mathbb{E}\left[\sum_{j=0}^{\infty} \beta^j y_{t+j} \middle| x_t\right] \\
&= G(I - \beta A)^{-1} x_t
\end{aligned}
$$

- Relabel Markov Chains to match the algebra: $x \equiv \pi^\top, A \equiv P^\top, C = 0$

Expected Present Discounted Value

- Consider an asset with period payoffs in $x_1, \dots x_N$ with transitions according to P
- Risk-neutral **expected present discounted value(EPDV)**

$$
\begin{aligned}
p(X_t) &= \mathbb{E}\left[\sum_{j=0}^{N} \beta^j X_{t+j} \,\middle|\, X_t\right] \\
&= \sum_{j=0}^{N} \beta^j \underbrace{\mathbb{E}\left[X_{t+j} \,\middle|\, X_t\right]}_{=G(\pi_t P^j)^\top} \\
&= G(I - \beta P^\top)^{-1} \pi_t^\top
\end{aligned}
$$

→ Note the connection to the LSS

Stationarity and Ergodicity

Stationary Distribution

- Take some X_t initial condition, does this converge?

$$\lim_{j\to\infty} X_{t+j} \,|\, X_t = \lim_{j\to\infty} \pi_t \cdot P^j = \pi_\infty ?$$

  → Does it exist? Is it unique?
- How does it compare to fixed point below, i.e. does $\pi^* = \pi_\infty$ for all X_t?

$$\pi^* = \pi^* \cdot P$$

  → This is the eigenvector associated with the eigenvalue of 1 of $P^\top$

  → Can prove there is always at least one. If more than one, multiplicity

Stochastic Matrices

- P is a **stochastic matrix** if
  - → $\sum_{j=1}^{N} P_{ij} = 1$ for all i, e.g. rows are conditional distributions
- **Key Properties:**
  - → One (or more) eigenvalue of 1 with associated left-eigenvector π

$$\pi P = \pi$$

  - → Equivalently the right eigenvector with eigenvalue $= 1$

$$P^\top \pi^\top = 1 \times \pi^\top$$

  - → Where we can normalize to $\sum_{n=1}^{N} \pi_i = 1$

Calculating Stationary Distributions

- Compare the steady states
  - → Left-eigenvector: $\pi^* = \pi^* P$ (calculate with right-eigenvector $\mathbf{1} \times \pi^{*\top} = P^\top \pi^{*\top}$)
  - → Limiting distribution: $\lim_{T\to\infty} \pi_0 P^T$
- Can show that the stationary distribution is $\pi^* = \begin{bmatrix} \frac{\lambda}{\alpha+\lambda} & \frac{\alpha}{\alpha+\lambda} \end{bmatrix}$

```julia
eigvals, eigvecs = eigen(P')
index = findfirst(x -> isapprox(x, 1), eigvals)
pi_star = real.(vec(eigvecs[:, index]))
pi_star = pi_star / sum(pi_star)
pi_0 = [1.0, 0.0]
pi_inf = pi_0' * (P^100) # \approx infty?
println("pi_star = ", pi_star)
println("pi_inf = ", pi_inf);
```

```
pi_star = [0.6666666666666666, 0.3333333333333333]
pi_inf = [0.6666666666666629 0.33333333333333154]
```

Communicating States

- Consider two states X_i and X_j ordered by indices i and j in P,
- If it is possible to move from X_i to X_j in a finite number of steps, the states are said to **communicate**
- Formally, X_i and Y_j communicate if there exist l and m such that

$$P_{ij}^{l} > 0 \quad \text{and} \quad P_{ji}^{m} > 0$$

→ Consider transition probabilities to see why this implies communication

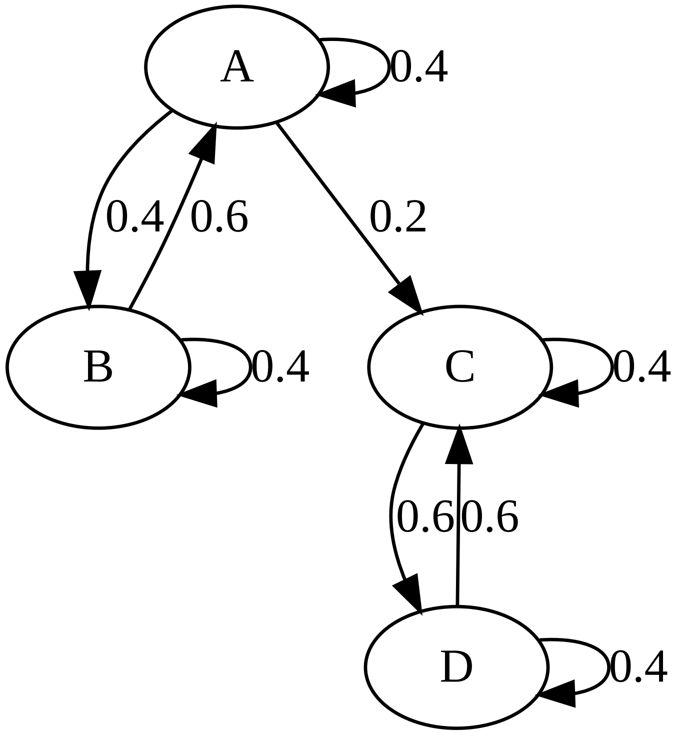
Irreducibility

- A Markov chain is **irreducible** if all states communicate with each other
- Calculated in practice with tools such as strongly connected components from Graph Theory

```
mc = MarkovChain(P)
@show is_irreducible(mc);
```

```
is_irreducible(mc) = true
```

Example: Not-Irreducible




```
P2 = [0.4 0.4 0.2 0.0;
      0.6 0.4 0.0 0.0;
      0.0 0.0 0.4 0.6;
      0.0 0.0 0.6 0.4]
mc2 = MarkovChain(P2)
@show is_irreducible(mc2);
```

```
is_irreducible(mc2) = false
```

Periodicity

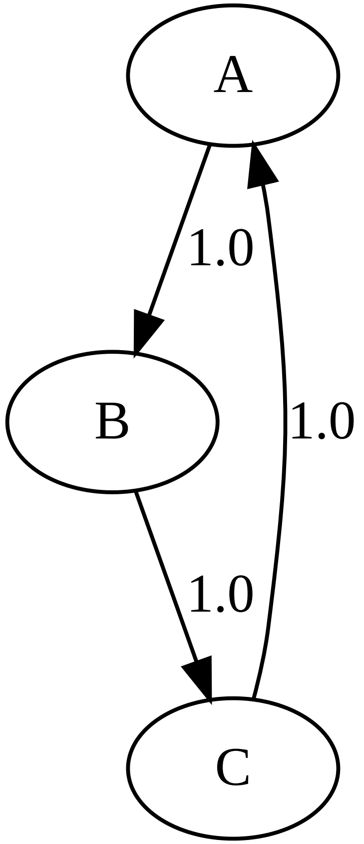
- Loosely speaking, a Markov chain is called periodic if it cycles in a predictable way, and aperiodic otherwise
- See **here** for more details

  → The "period" is the greatest common divisor of the set of times at which the chain can return to a state

```
mc = MarkovChain(P)
@show is_aperiodic(mc);
```

```
is_aperiodic(mc) = true
```

Example: Aperiodic




```
P3 = [0 1 0; 0 0 1; 1 0 0]
mc3 = MarkovChain(P3)
@show is_aperiodic(mc3);
```

```
is_aperiodic(mc3) = false
```

Theorems for Stationarity

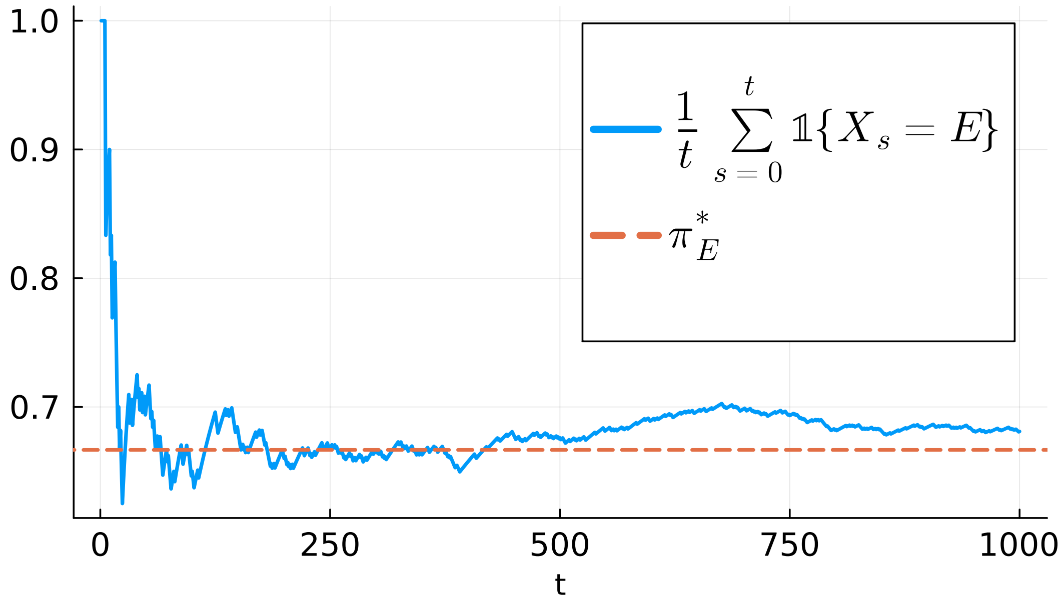
- **Theorem** Every stochastic matrix P has at least one stationary distribution.
- **Theorem** If P is irreducible and aperiodic then
  - → it has a unique stationary distribution π^*
  - → for any initial distribution π_0, $\lim_{T\to\infty} \pi_0 P^T = \pi^*$
  - → $P_{ij} > 0$ for all i, j is a sufficient condition
  - → it is **ergodic**. With $\mathbb{1}\{\cdot\}$ the indicator function

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{1}\{X_t = x_i\} = \pi_i^*, \quad \text{for all } i$$

Ergodicity

- These is the same sense of **ergodicity** we discussed before

```julia
alpha, lambda = 0.3, 0.6
P = [1-alpha alpha; lambda 1-lambda]
mc = MarkovChain(P)
pi_star = stationary_distributions(mc)[1]
T = 1000
init=1
X = simulate(mc, T;init)
prop_E_t = cumsum(X.==1)./(1:length(X))
plot(1:T, prop_E_t, xlabel="t",
 label=L"\frac{1}{t}\sum_{s=0}^t \mathbb{1}\{X_s = E\}"
 size=(600, 400))
hline!([pi_star[1]]; label=L"\pi^{*}_E",
linestyle=:dash)
```



Discretizing Continuous State Processes

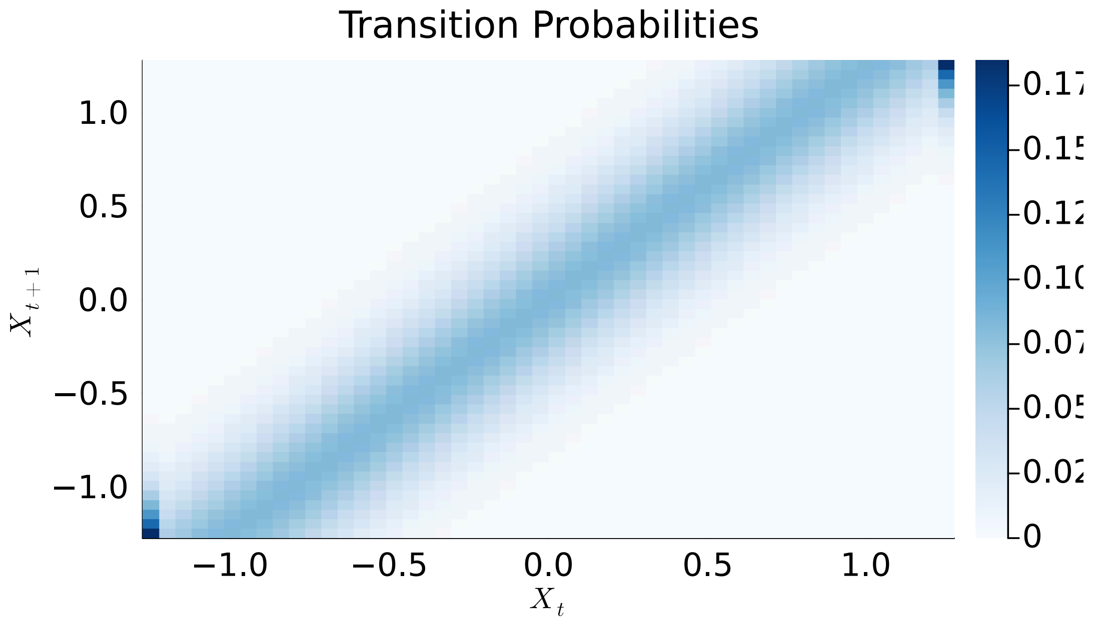
Discretization

- Unless continuous variables are easily summarized by a finite number of parameters or statistics, we will need to convert continuous functions and stochastic processes into discrete ones.
- Hence, to implement many algorithms, it is useful to model decisions with a finite number of states
  - → If the natural stochastic process is discrete, then no problem
  - → Otherwise, you can **discretize** the continuous time process into N states
  - → Try to ensure crucial statistics are preserved
  - → N might be very large!

AR(1) Transition Probabilities

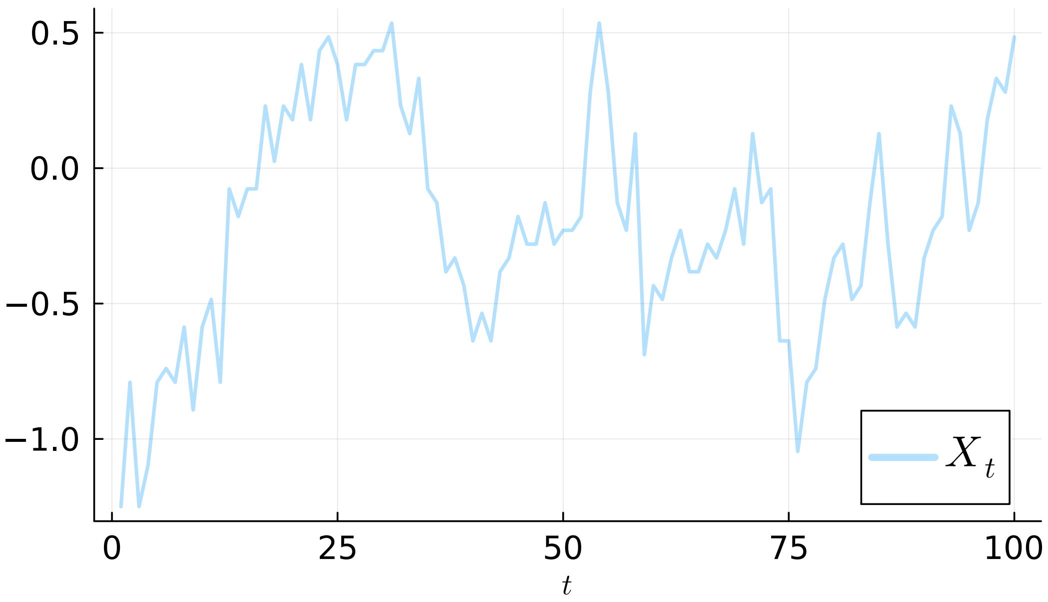
e.g. $X_{t+1} = \rho X_t + \sigma w_{t+1}$ using Tauchen's Method

```julia
N = 50 # number of nodes
rho = 0.8
sigma = 0.25
mc = tauchen(N, rho, sigma)
X_vals = mc.state_values
heatmap(X_vals, X_vals, mc.p;
        xlabel=L"X_t",
        ylabel=L"X_{t+1}",
        title="Transition Probabilities",
        color=:Blues,
        size=(600, 400))
```



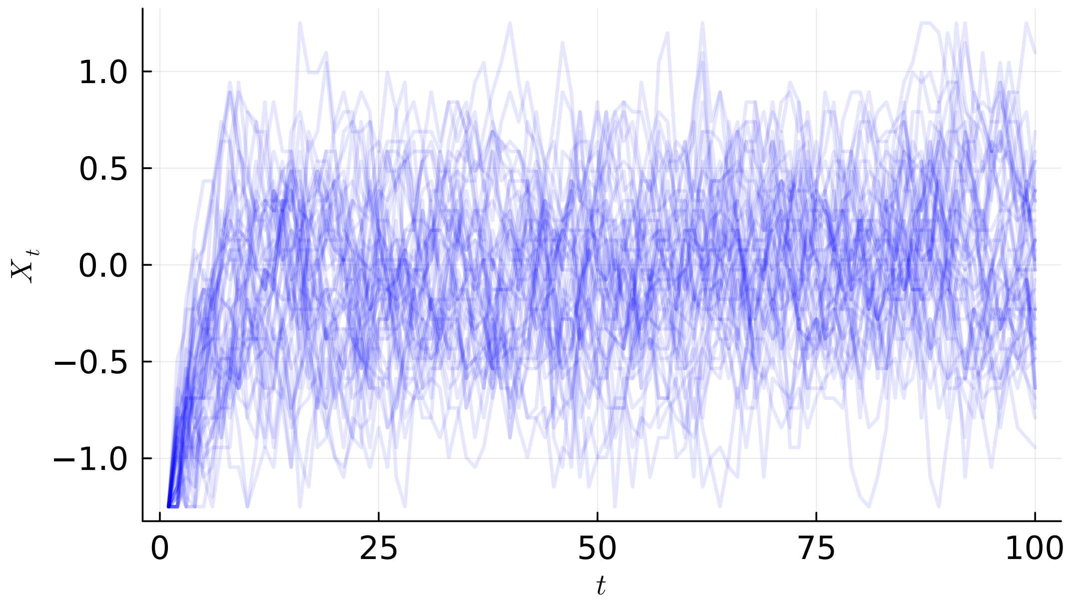
Simulation

```
T = 100
X = simulate(mc, T;init=1)
plot(X, xlabel=L"t", label=L"X_t",
     alpha = 0.3, size=(600, 400))
```



Ensemble

```julia
T = 100
num_chains = 50
plt = plot(;ylabel=L"X_t", xlabel=L"t",
            size=(600, 400), legend=false)
for i in 1:num_chains
    X = simulate(mc, T;init=1)
    plot!(X; alpha = 0.1, color=:blue)
end
plt
```



Lake Model of Unemployment and Employment

Individual Worker

- Consider a worker who can be either employed (E) or unemployed (U), following our previous markov chain
- Assign the value of 0 if unemployed and 1 if employed
- Lets calculate the cumulative proportion of their time employed

Reminder on Long-Run

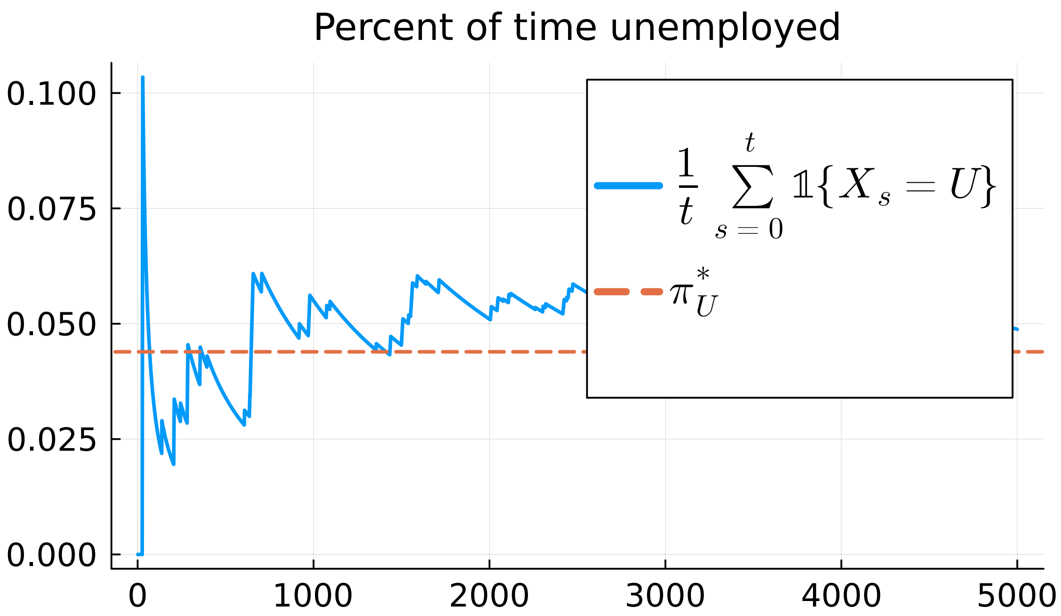
- What is the probability in the distant future of being employed?
- Note ergodic interpretation!

```julia
lambda = 0.283
alpha = 0.013
T = 5000
# order U, E
P = [1-lambda lambda; alpha 1-alpha]
mc = MarkovChain(P)
@show stationary_distributions(mc)[1]
eigvals, eigvecs = eigen(P')
index = findfirst(x -> isapprox(x, 1), eigvals)
pi_star = real.(vec(eigvecs[:, index]))
pi_star = pi_star / sum(pi_star)
@show pi_star;
```

```
(stationary_distributions(mc))[1] = [0.043918918918918914,
0.956081081081081]
pi_star = [0.04391891891891895, 0.9560810810810811]
```

Cumulative Employment

```julia
mc = MarkovChain(P, [0; 1])  # U -> 0, E -> 1
s_path = simulate(mc, T; init = 2)
u_bar, e_bar = stationary_distributions(mc)[1]
# Note mapping in MarkovChain
s_bar_e = cumsum(s_path) ./ (1:T)
s_bar_u = 1 .- s_bar_e
s_bars = [s_bar_u s_bar_e]
plot(title = "Percent of time unemployed",
 1:T, s_bars[:, 1], lw = 2,
 label=L"\frac{1}{t}\sum_{s=0}^t \mathbb{1}\{X_s = U\}"
 legend=:topright, size=(600, 400))
hline!([u_bar], linestyle = :dash,
       label = L"\pi^{*}_U")
```



Many Workers

- Consider if an entire economy is populated by workers of these types
- With approximately a continuum of agents of this type, can we interpret the statistical distribution of the states as a fraction in the distribution?
- This is a key trick used throughout macro, but is subtle
- We will assume a continuum of agents, but add in:
  - → A proportion d die each period
  - → A proportion b are born each period (into the U state)
  - → Define $g \equiv b - d$, the net growth rate

Definitions

- To track distributions, a tight connection to the "adjoint" of the stochastic process for the Markov Chain
- Instead, building it directly from flows, define
  - → E_t, the total number of employed workers at date t
  - → U_t, the total number of unemployed workers at t
  - → N_t, the number of workers in the labor force at t
  - → The employment rate $e_t \equiv E_t/N_t$.
  - → The unemployment rate $u_t \equiv U_t/N_t$.

Laws of Motion for Stock Variables

- Of the mass of workers E_t who are employed at date t,
  - → $(1-d)E_t$ remain in N_t, and $(1-\alpha)(1-d)E_t$ remain in E_t

$$E_{t+1} = (1-d)(1-\alpha)E_t + (1-d)\lambda U_t$$

- Of the mass of workers U_t workers who are currently unemployed,
  - → $(1-d)U_t$ will remain in N_t and $(1-d)\lambda U_t$ enter E_t

$$U_{t+1} = (1-d)\alpha E_t + (1-d)(1-\lambda)U_t + b(E_t + U_t)$$

- The total stock of workers $N_t = E_t + U_t$ evolves as

$$N_{t+1} = (1+b-d)N_t = (1+g)N_t$$

Summarizing

- Letting $\boldsymbol{X}_t \equiv \begin{bmatrix} U_t \\ E_t \end{bmatrix}$, the law of motion for $\boldsymbol{X}$ is

$$X_{t+1} = \underbrace{\begin{bmatrix} (1-d)(1-\lambda)+b & (1-d)\alpha + b \\ (1-d)\lambda & (1-d)(1-\alpha) \end{bmatrix}}_{\equiv A} X_t$$

  → Note: $\boldsymbol{A} = (1-d)P^\top + \begin{bmatrix} b & b \\ 0 & 0 \end{bmatrix}$

  → Take a class in stochastic processes!

Laws of Motion for Rates

- Define $\boldsymbol{x}_t \equiv \begin{bmatrix} u_t \\ e_t \end{bmatrix} = \begin{bmatrix} U_t/N_t \\ E_t/N_t \end{bmatrix}$
- Divide both sides of $X_{t+1} = AX_t$ by N_{t+1} and simplify to get

$$x_{t+1} = \underbrace{\frac{1}{1+g}A}_{\equiv \hat{A}} x_t$$

→ You can check that $e_t + u_t = 1$ implies that $e_{t+1} + u_{t+1} = 1$

Longrun Distribution

- To find the long-run distribution of employment rates note,

$$x^* = \hat{A}x^* = h(x^*)$$

  - → So could ,find a fixed point of $h(\cdot)$
  - → Or solve an eigenvalue problem.
- Note that if $g \neq \mathbf{0}$, there is no fixed point of $X_{t+1} = AX_t$

Reminder: Simple Function Iteration

```julia
function iterate_map(f, x0, T)
    x = zeros(length(x0), T + 1)
    x[:, 1] = x0
    for t in 2:(T + 1)
        x[:, t] = f(x[:, t - 1])
    end
    return x
end
```

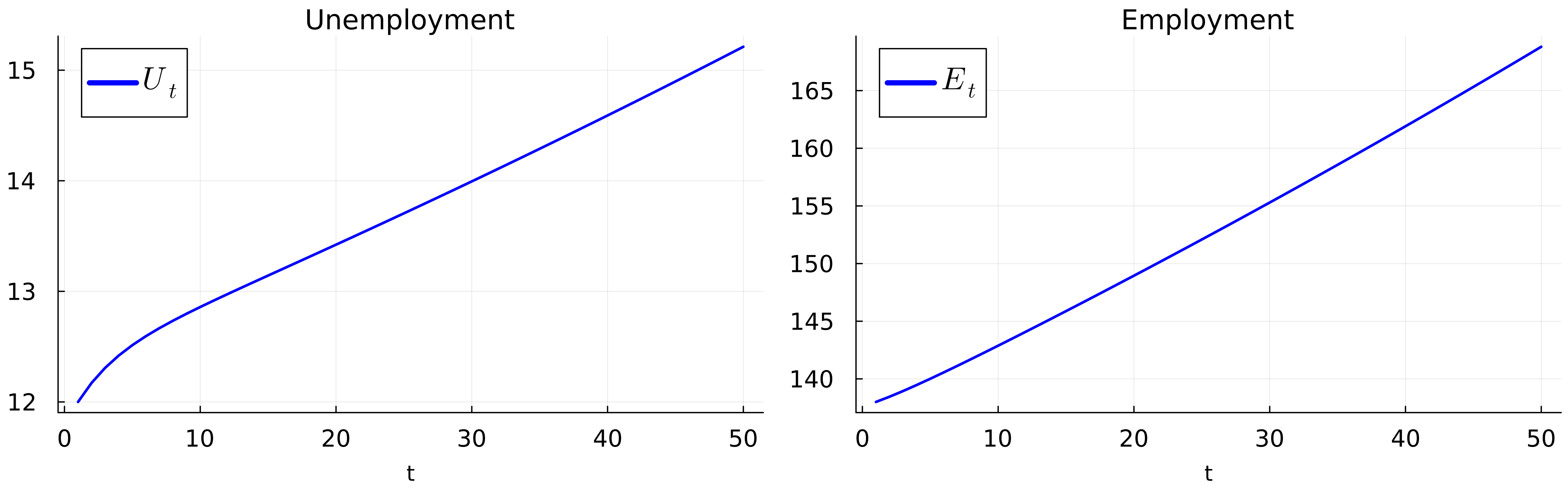
Implementation of a Lake Model

```julia
function lake_model(; lambda = 0.283, alpha = 0.013, b = 0.0124, d = 0.00822)
    g = b - d
    A = [(1 - lambda) * (1 - d)+b (1 - d) * alpha+b
         (1 - d)*lambda (1 - d)*(1 - alpha)]
    A_hat = A ./ (1 + g)
    x_0 = ones(size(A_hat, 1)) / size(A_hat, 1)
    sol = fixedpoint(x -> A_hat * x, x_0)
    converged(sol) || error("Failed to converge in $(sol.iterations) iter")
    x_bar =sol.zero
    return (; lambda, alpha, b, d, A, A_hat, x_bar)
end
```

Aggregate Dynamics

```
lm = lake_model()
N_0 = 150
e_0 = 0.92
u_0 = 1 - e_0
T = 50
U_0 = u_0 * N_0
E_0 = e_0 * N_0
X_0 = [U_0; E_0]
X_path = iterate_map(X -> lm.A * X, X_0, T - 1)
x1 = X_path[1, :]
x2 = X_path[2, :]
plt_unemp = plot(1:T, X_path[1, :]; color = :blue,
                 label = L"U_t", xlabel="t", title = "Unemployment")
plt_emp = plot(1:T, X_path[2, :]; color = :blue,
               label = L"E_t", xlabel="t", title = "Employment")
plot(plt_unemp, plt_emp, layout = (1, 2), size = (1200, 400))
```

Aggregate Dynamics



Transitions of Rates

```
u_bar, e_bar = lm.x_bar
x_0 = [u_0; e_0]
x_path = iterate_map(x -> lm.A_hat * x, x_0, T - 1)
plt_unemp = plot(1:T, x_path[1, :];title = "Unemployment rate",
                 color = :blue, label = L"u_t")
hline!(plt_unemp, [u_bar], color = :red, linestyle = :dash, label = L"\pi^{*}_U")
plt_emp = plot(1:T, x_path[2, :]; title = "Employment rate", color = :blue, label = L"e_t")
hline!(plt_emp, [e_bar], color = :red, linestyle = :dash,label = L"\pi^{*}_E")
plot(plt_unemp, plt_emp, layout = (1, 2), size = (1200, 400))
```

Transitions of Rates