Geometric Series, Fixed Points, and Asset Pricing

Undergraduate Computational Macro

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Overview

Motivation and Materials

- In this lecture, we will introduce **fixed points**, practice a little Julia coding, move on to **geometric series**
- The applications will be to **asset pricing** and **Keynesian multipliers**
  - → Asset pricing, in particular, will be something we come back to repeatedly as a way to practice our tools
- Even for those not interested in finance, you will see that many problems are tightly related to asset pricing
  - → Human capital accumulation, choosing when to accept jobs, etc.

Materials

- Adapted from QuantEcon lectures coauthored with John Stachurski and Thomas J. Sargent
  - → Julia by Example
  - → Geometric Series for Elementary Economics

```julia
using LinearAlgebra, Statistics, Plots, Random, Distributions, LaTeXStrings
default(;legendfontsize=16)
```

Intro to Fixed Points

Fixed Points

- Fixed points are everywhere!
  - → Lets first look at the mechanics and practice code, then apply them.
- Take a mapping $f : X \to X$ for some set X.
  - → If there exists an $x^* \in X$ such that $f(x^*) = x^*$, then x^*: is called a "fixed point" of f
- A fixed point is a property of a function, and may not be unique
- Lets walk through the math, and then practice a little more Julia coding with them

Simple, Linear Example

- For given scalars y, β and a scalar v of interest

$$v = y + \beta v$$

- If $|\beta| < 1$, then this can can be solved in closed form as $v = y/(1-\beta)$
- Rearrange the equation in terms of a map $f : \mathbb{R} \to \mathbb{R}$

$$f(v) := y + \beta v$$

- Therefore, a fixed point $f(\cdot)$ is a solution to the above problem such that $v = f(v)$

Fixed Point Iteration

- Consider iteration of the map $\boldsymbol{f}$ starting from an initial condition $\boldsymbol{v}_0$

$$v_{n+1} = f(v_n)$$

- Does this converge? Depends on $\boldsymbol{f}(\cdot)$, as we will explore in detail
  → It shouldn't depend on $\boldsymbol{v}_0$ or there is an issue
- See Banach's fixed-point theorem

When to Stop Iterating?

- If v_n is a scalar, then we can check convergence by looking at $|v_{n+1} - v_n|$ with some threshold, which may be problem dependent
  - → If v_n will be a vector, so we should use a norm $||v_{n+1} - v_n||$
  - → e.g. the Euclidean norm, `norm(v_new - v_old)` in Julia
- Keep numerical precision in mind! Can see this in Julia with the following

```julia
@show eps() #machine epsilon, the smallest number such that 1.0 + eps() > 1.0
@show 1.0 + eps()/2 > 1.0;
```

```
eps() = 2.220446049250313e-16
1.0 + eps() / 2 > 1.0 = false
```

Verifying with the Linear Example

- For our simple linear map: $f(v) \equiv y + \beta v$
- Iteration becomes $v_{n+1} = y + \beta v_n$. Iterating backwards

$$v_{n+1} = y + \beta v_n = y + \beta y + \beta^2 v_{n-1} = y \sum_{i=0}^{n-1} \beta^i + \beta^n v_0$$

→ $\sum_{i=0}^{n-1} \beta^i = \frac{1-\beta^n}{1-\beta}$ and $\sum_{i=0}^{\infty} \beta^i = \frac{1}{1-\beta}$ if $|\beta| < 1$

→ So $n \to \infty$, converges to $v = y/(1-\beta)$ for all v_0

Implementing with For Loop

```julia
y = 1.0
beta = 0.9
v_iv = 0.8 # initial condition
v_old = v_iv
normdiff = Inf
iter = 1
for i in 1:1000
    v_new = y + beta * v_old # the f(v) map
    normdiff = norm(v_new - v_old)
    if normdiff < 1.0E-7 # check convergence
        iter = i
        break # converged, exit loop
    end
    v_old = v_new # replace and continue
end
println("Fixed point = $v_old  |f(x) - x| = $normdiff in $iter iterations");
```

```
Fixed point = 9.999999081896231  |f(x) - x| = 9.181037796679448e-8 in 154 iterations
```

Implementing in Julia with While Loop

```julia
v_old = v_iv
normdiff = Inf
iter = 1
while normdiff > 1.0E-7 && iter <= 1000
    v_new = y + beta * v_old # the f(v) map
    normdiff = norm(v_new - v_old)
    v_old = v_new # replace and continue
    iter = iter + 1
end
println("Fixed point = $v_old  |f(x) - x| = $normdiff in $iter iterations")
```

```
Fixed point = 9.999999173706609  |f(x) - x| = 9.181037796679448e-8 in 155 iterations
```

Avoid Global Variables

```julia
function v_fp(beta, y, v_iv; tolerance = 1.0E-7, maxiter=1000)
    v_old = v_iv
    normdiff = Inf
    iter = 1
    while normdiff > tolerance && iter <= maxiter
        v_new = y + beta * v_old # the f(v) map
        normdiff = norm(v_new - v_old)
        v_old = v_new
        iter = iter + 1
    end
    return (v_old, normdiff, iter) # returns a tuple
end
y = 1.0
beta = 0.9
v_star, normdiff, iter = v_fp(beta, y, 0.8)
println("Fixed point = $v_star |f(x) - x| = $normdiff in $iter iterations")
```

```
Fixed point = 9.999999173706609 |f(x) - x| = 9.181037796679448e-8 in 155 iterations
```

Use a Higher Order Function and Named Tuple

- Why hardcode the mapping? Pass it in as a function
- Lets add in keyword arguments and use a named tuple for clarity

```julia
function fixedpointmap(f, iv; tolerance = 1.0E-7, maxiter=1000)
    x_old = iv
    normdiff = Inf
    iter = 1
    while normdiff > tolerance && iter <= maxiter
        x_new = f(x_old) # use the passed in map
        normdiff = norm(x_new - x_old)
        x_old = x_new
        iter = iter + 1
    end
    return (; value = x_old, normdiff, iter) # A named tuple
end
```

```
fixedpointmap (generic function with 1 method)
```

Passing in a Function

```
y = 1.0
beta = 0.9
v_initial = 0.8
f(v) = y + beta * v # note that y and beta are used in the function!
sol = fixedpointmap(f, 0.8; tolerance = 1.0E-8) # don't need to pass
println("Fixed point = $(sol.value) |f(x) - x| = $(sol.normdiff) in $(sol.iter) iterations")

# Unpacking notation for the named tuples not sensitive to order
(; value, iter, normdiff) = fixedpointmap(v -> y + beta * v, # creates an anonymous "closure"
                                          v_initial; tolerance = 1.0E-8)
println("Fixed point = $value |f(x) - x| = $normdiff in $iter iterations")
```

```
Fixed point = 9.999999918629035 |f(x) - x| = 9.041219328764782e-9 in 177 iterations
Fixed point = 9.999999918629035 |f(x) - x| = 9.041219328764782e-9 in 177 iterations
```

Other Algorithms

- VFI is instructive, but not always the fastest
- Can also write as a "root finding" problem
  - → i.e. $\hat{\boldsymbol{f}}(\boldsymbol{x}) \equiv \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{x}$ so that $\hat{\boldsymbol{f}}(\boldsymbol{x}^*) = \boldsymbol{0}$ is the fixed point
  - → These can be especially fast if $\nabla \hat{\boldsymbol{f}}(\cdot)$ is available
- Another is called Anderson Acceleration
  - → The fixed-point iteration we have above is a special case

Use Packages with Better Algorithms

- NLsolve.jl has equations for solving equations (and fixed points)
  - → e.g., 3 iterations, not 177, for Andersen Acceleration
- Uses multi-dimensional maps, so can write in that way rather than scalar

```julia
using NLsolve
# best style
y = 1.0
beta = 0.9
iv = [0.8] # note move to array
f(v) = y .+ beta * v # note that y and beta are used in the function!
sol = fixedpoint(f, iv) # uses Anderson Acceleration
fnorm = norm(f(sol.zero) .- sol.zero)
println("Fixed point = $(sol.zero) |f(x) - x| = $fnorm  in $(sol.iterations) iterations")
```

```
Fixed point = [9.999999999999972] |f(x) - x| = 3.552713678800501e-15  in 3 iterations
```

Geometric Series and PDVs

Geometric Series

- Finite geometric series

$$1 + c + c^2 + c^3 + \cdots + c^T = \sum_{t=0}^{T} c^t = \frac{1 - c^{T+1}}{1 - c}$$

- Infinite geometric series, requiring $|c| < 1$

$$1 + c + c^2 + c^3 + \cdots = \sum_{t=0}^{\infty} c^t = \frac{1}{1 - c}$$

Discounting

- In discrete time, $t = 0, 1, 2, \ldots$
- Let $r > 0$ be a one-period **net nominal interest rate**
- A one-period **gross nominal interest rate** R is defined as

$$R = 1 + r > 1$$

- If the nominal interest rate is 5 percent, then $r = 0.05$ and $R = 1.05$

Interpretation as Prices

- The gross nominal interest rate R is an **exchange rate** or **relative price** of dollars at between times t and $t+1$. The units of R are dollars at time $t+1$ per dollar at time t.
- When people borrow and lend, they trade dollars now for dollars later or dollars later for dollars now.
- The price at which these exchanges occur is the gross nominal interest rate.
  - → If I sell x dollars to you today, you pay me Rx dollars tomorrow.
  - → This means that you borrowed x dollars for me at a gross interest rate R and a net interest rate r.
- In equilibrium, the prices for borrowing and lending should be related

Where do Interest Rates Come From?

- More later, but consider connection to a discount factor $\beta \in (0, 1)$ in **consumer preferences**
- This represents how much consumers value future consumption tomorrow relative to today
- In some simple cases $R^{-1} = \beta$ makes sense
  - → Much more later, including how to think about cases with randomness
- For now, just use R^{-1} directly as a discount factor, thinking about risk-neutrality

Accumulation

- $x, xR, xR^2, \cdots$ tells us how investment of x dollar value of an investment **accumulate** through time. Compounding
- Reinvested in the project (i.e., compounding)
  - → thus, 1 dollar invested at time 0 pays interest r dollars after one period, so we have $r + 1 = R$ dollars at time 1
  - → at time 1 we reinvest $1 + r = R$ dollars and receive interest of rR dollars at time 2 plus the **principal** R dollars, so we receive $rR + R = (1 + r)R = R^2$ dollars at the end of period 2

Discounting

- $1, R^{-1}, R^{-2}, \cdots$ tells us how to **discount** future dollars to get their values in terms of today's dollars.
- Tells us how much future dollars are worth in terms of today's dollars.
- Remember that the units of R are dollars at $t+1$ per dollar at t.
  - → the units of R^{-1} are dollars at t per dollar at $t+1$
  - → the units of R^{-2} are dollars at t per dollar at $t+2$
  - → and so on; the units of R^{-j} are dollars at t per dollar at $t+j$

Asset Pricing

- An asset has payments stream of y_t dollars at times $t = 0, 1, 2, \dots, G \equiv 1 + g, g > 0$ and $G < R \equiv 1 + r$

$$y_t = G^t y_0$$

  - → i.e. grows at g percent, discounted at r percent
- The **present value** of the asset is

$$p_0 = y_0 + y_1/R + y_2/(R^2) + \cdots = \sum_{t=0}^{\infty} y_t (1/R)^t = \sum_{t=0}^{\infty} y_0 G^t (1/R)^t$$

$$= \sum_{t=0}^{\infty} y_0 (G/R)^t = \frac{y_0}{1 - G/R}$$

Gordon Formula

- For small r and g, use a Taylor series or $rg \approx 0$ to get

$$GR^{-1} \approx 1 + g - r$$

- Hence,

$$p_0 = y_0/(1 - (1+g)/(1+r)) \approx \frac{y_0}{r-g}$$

Assets with Finite Lives

- Consider an asset that pays $y_t = 0$ for $t > T$ and $y_t = G^t y_0$ for $t \leq T$
  - → i.e., the same process but truncated it T periods
- The present value is

$$p_0 = \sum_{t=0}^{T} y_t (1/R)^t = \sum_{t=0}^{T} y_0 G^t (1/R)^t$$
$$= \sum_{t=0}^{T} y_0 (G/R)^t = y_0 \frac{1-(G/R)^{T+1}}{1-G/R}$$

- How large is $(G/R)^{T+1}$?
  - → If small, then infinite horizon may be a good approximation

Is Infinite Horizon a Reasonable Approximation?

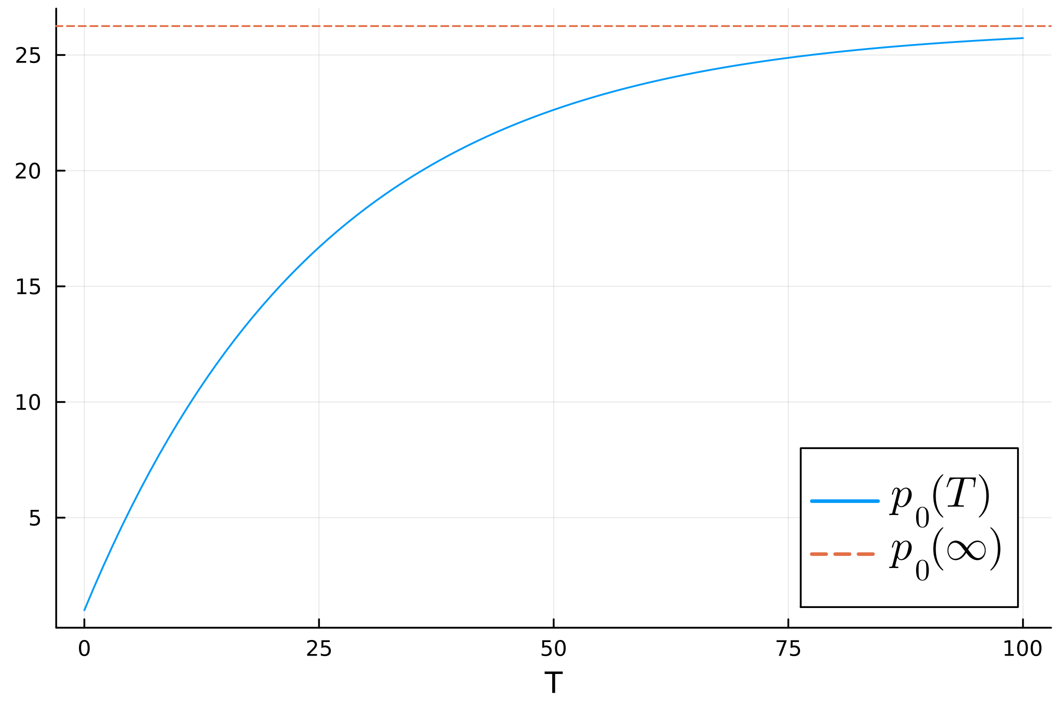
- Implement these in code to compare

```julia
infinite_payoffs(g, r, y_0) = y_0 / (1 - (1 + g) * (1 + r)^(-1))
function finite_payoffs(T, g, r, y_0)
    G = 1 + g
    R = 1 + r
    return (y_0 * (1 - G^(T + 1) * R^(-T - 1))) / (1 - G * R^(-1))
end
@show infinite_payoffs(0.01, 0.05, 1.0)
@show finite_payoffs(100, 0.01, 0.05, 1.0);
```

```
infinite_payoffs(0.01, 0.05, 1.0) = 26.249999999999994
finite_payoffs(100, 0.01, 0.05, 1.0) = 25.73063957477331
```

Comparing Different Horizons

```julia
g = 0.01
r = 0.05
y_0 = 1.0
T = 100
# broadcast over 0:T
p_finite = finite_payoffs.(0:T, g, r, y_0)
p_infinite = infinite_payoffs(g, r, y_0)
plot(0:T, p_finite,xlabel = "T",
     label= L"p_0(T)", size = (600,400))
hline!([p_infinite], linestyle = :dash,
       label = L"p_0(\infty)")
```

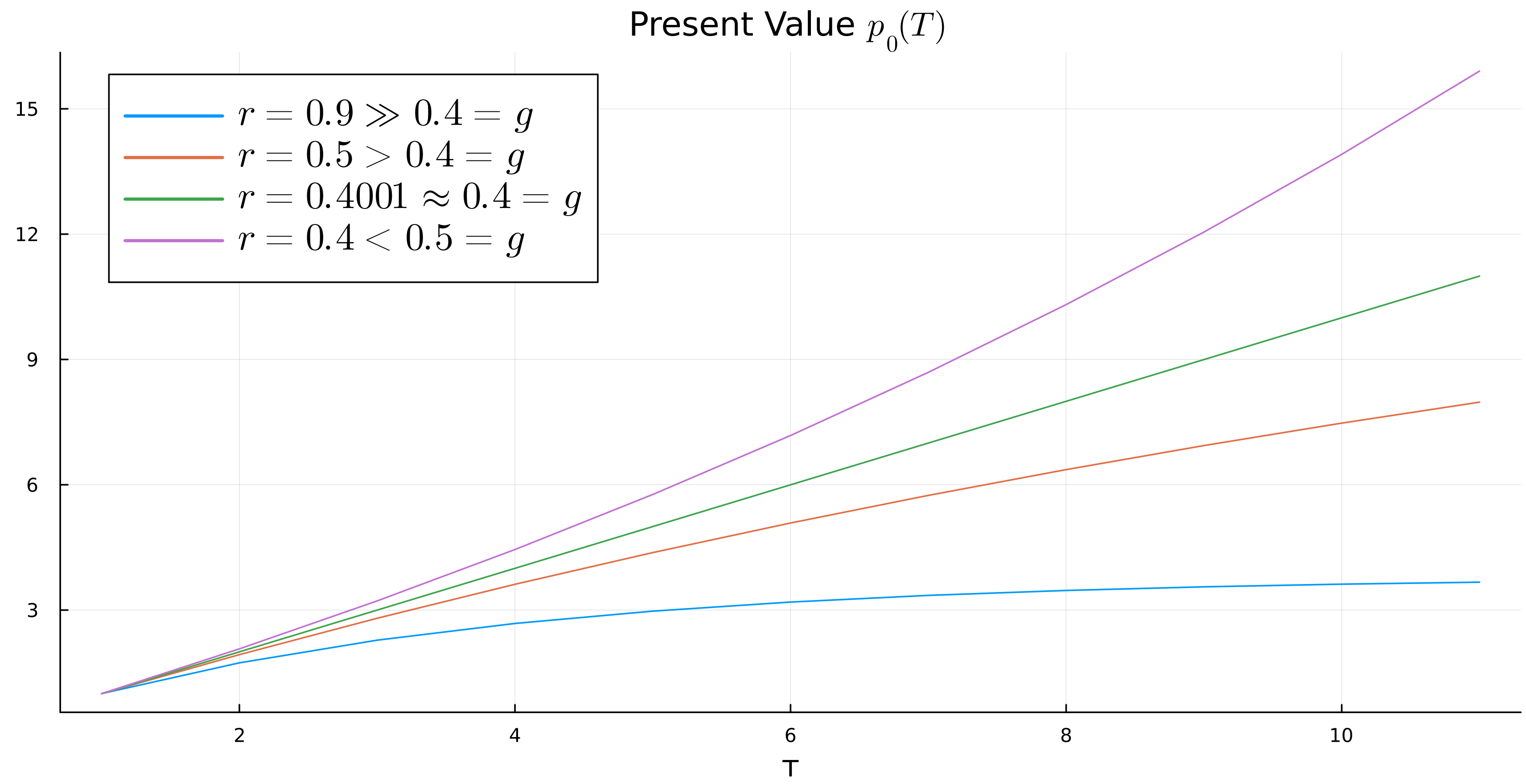


Discounting vs. Growth

- For $T = \infty$, we assumed that $GR^{-1} < 1$, or approximately $g < r$

```julia
T = 10
y_0 = 1.0
plot(title = L"Present Value $p_0(T)$", legend = :topleft, xlabel = "T")
plot!(finite_payoffs.(0:T, 0.4, 0.9, y_0),
      label = L"r=0.9 \gg 0.4 = g")
plot!(finite_payoffs.(0:T, 0.4, 0.5, y_0), label = L"r=0.5 > 0.4 = g")
plot!(finite_payoffs.(0:T, 0.4, 0.4001, y_0),
      label = L"r=0.4001 \approx 0.4 = g")
plot!(finite_payoffs.(0:T, 0.5, 0.4, y_0), label = L"r=0.4 < 0.5 = g")
```

Discounting vs. Growth



Asset Pricing and Fixed Points

Rewriting our Problem

- Lets write a version of the model for arbitrary y_t and relabel $\beta \equiv 1/R$
- The asset price, p_t starting at any t

$$p_t = \sum_{j=0}^{\infty} \beta^j y_{t+j}$$

$$\begin{aligned}
p_t &= y_t + \beta y_{t+1} + \beta^2 y_{t+2} + \beta^3 y_{t+3} + \cdots \\
&= y_t + \beta \left(y_{t+1} + \beta y_{t+2} + \beta^2 y_{t+2} \cdots\right) \\
&= y_t + \beta \sum_{j=0}^{\infty} \beta^j y_{t+j+1} \\
&= y_t + \beta p_{t+1}
\end{aligned}$$

Recursive Formulation

- In the simple case of $y_t = \bar{y}$, recursive equation is

$$p_t = \bar{y} + \beta p_{t+1}$$

  → We could also check that $p_t = \frac{\bar{y}}{1-\beta}$ fulfills this equation

  → There are be other p_t which fulfill it, but we won't explore that here

- In cases where the price is time-invariant, write this as a fixed point

$$p = \bar{y} + \beta p \equiv f(p)$$

Recursive Interpretation

$$p_t = y_t + \beta p_{t+1}$$

- The price p_t is the sum of
  - → The payoffs you get that period
  - → The discounted price of how much you can sell it next period
- The p_{t+1} is the **forecast** of the price tomorrow
  - → Here we are assuming the forecasts are perfect, as $\{y_t\}_{t=0}^{\infty}$ is known
- More generally, want expected price tomorrow using some probabilities

Solving Numerically

```julia
y_bar = 1.0
beta = 0.9
iv = [0.8]
f(p) = y_bar .+ beta * p
sol = fixedpoint(f, iv) # uses Anderson Acceleration
@show y_bar/(1 - beta), sol.zero;
```

```
(y_bar / (1 - beta), sol.zero) = (10.000000000000002, [9.999999999999972])
```

A More Complicated Example

- Instead $\bar{y}$, asset may pay y_L or y_H
  - → You don't know the payoff y_{t+1} until $t+1$ occurs
  - → You need to assign some probabilities of each occurring. e.g., equal
- As with the previous example, lets assume you hold onto the asset only a single period, then sell it
  - → Naturally, the value of the asset to both you and others depends on y_{t+1}
  - → We will see much more in future lectures
- Hint: in future lectures will use mathematical expectations

$$p_t = y_t + \beta \mathbb{E}\left[p_{t+1}\right]$$

Recursive Formulation

- Assume two prices: p_L and p_H for the asset depending on the y_t

$$
\begin{aligned}
p_L &= y_L + \beta\left[0.5 p_L + 0.5 p_H\right] \\
p_H &= y_H + \beta\left[0.5 p_L + 0.5 p_H\right]
\end{aligned}
$$

- Stack $p \equiv \begin{bmatrix} p_L & p_H \end{bmatrix}^\top$ and $y \equiv \begin{bmatrix} y_L & y_H \end{bmatrix}^\top$

$$
p = y + \beta \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} p \equiv f(p)
$$

  → We will see later how to write as a mathematical expectation

- We could solve this as a linear equation, but lets use a fixed point

Solving Numerically with a Fixed Point

```
y = [0.5, 1.5] #y_L, y_H
beta = 0.9
iv = [0.8, 0.8]
A = [0.5 0.5; 0.5 0.5]
sol = fixedpoint(p -> y .+ beta * A * p, iv) # f(p) := y + beta A p
p_L, p_H = sol.zero # can unpack a vector
@show p_L, p_H, sol.iterations
# p = y + beta A p =>  (I - beta A) p = y => p = (I - beta A)^{-1} y
@show (I - beta * A) \ y; # or $inv(I - beta * A) * y
```

```
(p_L, p_H, sol.iterations) = (9.500000000000028, 10.500000000000028, 4)
(I - beta * A) \ y = [9.499999999999996, 10.499999999999996]
```

Keynesian Multipliers

Model without Prices

- c: **consumption**, i: **investment**, g: **government expenditures**, y **national income**
- Prices don't adjust/exit to clear markets
  - → **Excess supply** of labor and capital (unemployment and unused capital)
  - → Prices and interest rates fail to adjust to make aggregate **supply equal demand** (e.g., prices and interest rates are frozen)
  - → National income entirely determined by aggregate demand, $\uparrow \boldsymbol{c} \implies \uparrow \boldsymbol{y}$

Simple Model

- **Assume**: consume a fixed fraction $0 < b < 1$ of the national income y_t
  - → b is the **marginal propensity to consume (MPC)**
  - → $1 - b$ is the **marginal propensity to save**
  - → Modern macro would have b adjust to reflect prices, consumer preferences, etc. and add in prices/production functions
- Leads to three equations in this basic model
  - → An accounting identity for the national income, the investment choice, and the consumer choice above

Equations

- **National income** is an accounting identity: the sum of consumption, investment, and government expenditures is the national income

$$y_t = c_t + i_t + g_t$$

- **Investment** private + government investment. Assume it is fixed here at i and g. Embeds behavioral assumptions?
- **Consumption** $c_t = by_{t-1}$, i.e. "behavior", not accounting. Lag on last periods income/output

Dynamics of Income and Consumption

- Substituting the consumption equation into the national income equation

$$
\begin{aligned}
y_t &= c_t + i + g \\
y_t &= b y_{t-1} + i + g \\
y_t &= b(b y_{t-2} + i + g) + i + g \\
y_t &= b^2 y_{t-2} + b(i+g) + (i+g)
\end{aligned}
$$

- Iterative backwards to a y_0,

$$
y_t = \sum_{j=0}^{t-1} b^j (i+g) + b^t y_0 = \frac{1-b^t}{1-b}(i+g) + b^t y_0
$$

Keynesian Multiplier

- Take limit as $t \to \infty$ to get

$$\lim_{t \to \infty} y_t = \frac{1}{1-b}(i+g)$$

- Define the **Keynesian multiplier** is $1/(1-b)$
  - → More consumption delivers higher income, which delivers more consumption, compounding...
  - → $i \to i + \Delta$ implies $y \to y + \Delta/(1-b)$. Same with g
- Is this correct (or useful) of a model?
  - → Probably not...gives intuition for more believable models
  - → Lets us practice difference equations

Iterating the Difference Equations

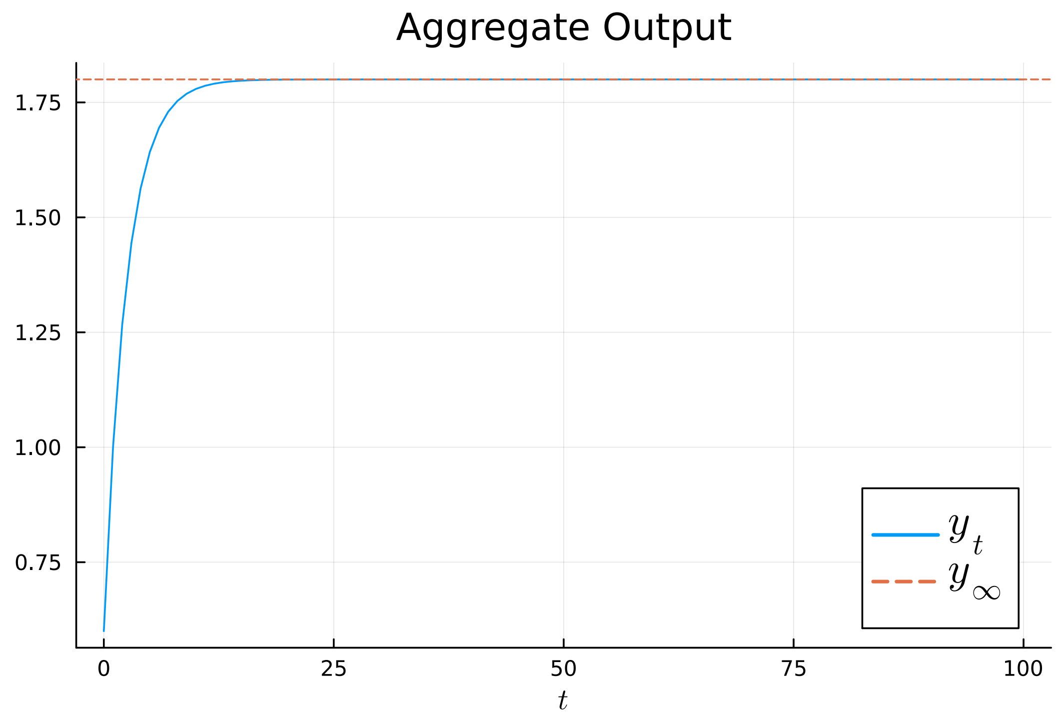
$$y_t = by_{t-1} + i + g$$

```
function calculate_y(i, b, g, T, y_0)
    y = zeros(T + 1)
    y[1] = i + b * y_0 + g
    for t in 2:(T + 1)
        y[t] = b * y[t - 1] + i + g
    end
    return y
end
y_limit(i, b, g) = (i + g) / (1 - b)
```

```
y_limit (generic function with 1 method)
```

Plotting Dynamics

```
i_0 = 0.3
g_0 = 0.3
b = 2/3 # = MPC out of income
y_0 = 0
T = 100
plot(0: T,calculate_y(i_0, b, g_0, T, y_0);
     title = "Aggregate Output",
     size=(600,400), xlabel = L"t",
     label = L"y_t")
hline!([y_limit(i_0, b, g_0)];
       linestyle = :dash,
       label = L"y_{\infty}")
```

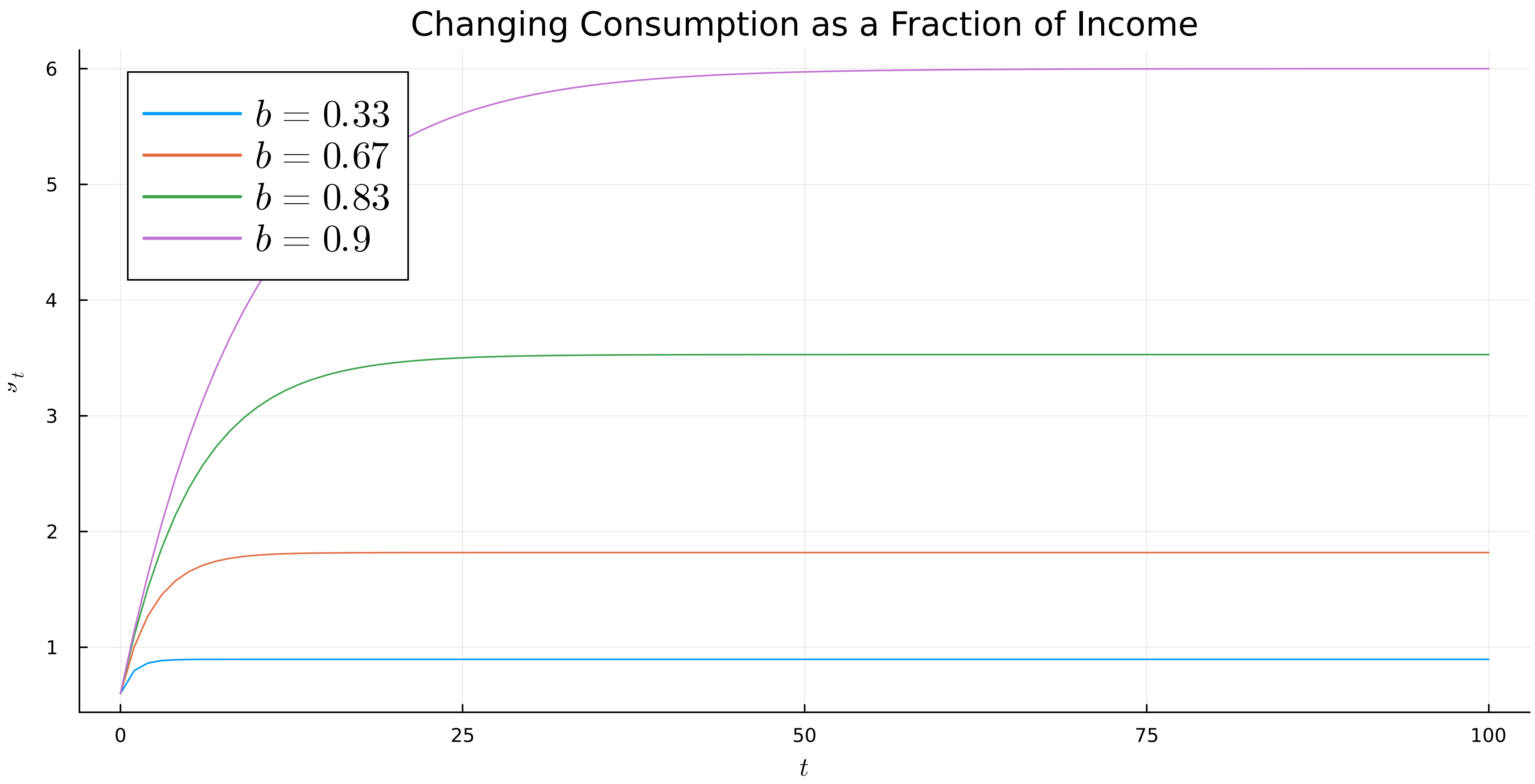


MPCs

- Suggests that national output, y_t is increasing in MPC, b, due to multiplier
- To increase the longrun size of economy, decrease the savings rate $(1 - b)$!

```julia
bs = round.([1 / 3, 2 / 3, 5 / 6, 0.9], digits = 2)
plt = plot(title = "Changing Consumption as a Fraction of Income",
           xlabel = L"t", ylabel = L"y_t", legend = :topleft)
[plot!(plt, 0:T, calculate_y(i_0, b, g_0, T, y_0), label = L"b = %$b")
 for b in bs]
plt
```

MPCs



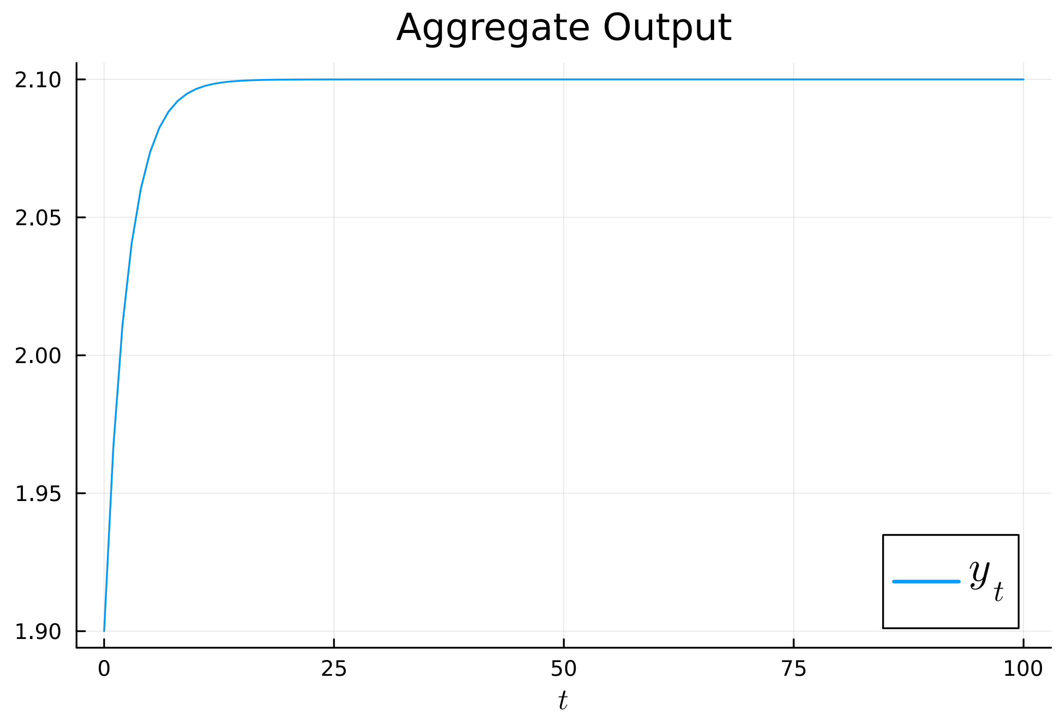
Changing Consumption as a Fraction of Income
$b = 0.33$
$b = 0.67$
$b = 0.83$
$b = 0.9$
6
5
4
3
2
1
y_t
0
25
50
75
100
t

Can Governments (Magically) Expand Output?

- Remember the limitation is that demand is too low and there is excess supply of labor and/or capital
- What if the government increases g by Δ?
  - → $y \rightarrow y + \Delta/(1-b)$
- Assume we start at the y_∞ for the $g = 0.3$
  - → Then we simulate dynamics for a permanent change to $g_1 = 0.4$

Plotting Dynamics for Government Intervention

```
y_lim = y_limit(i_0, b, g_0)
Delta_g = 0.1
y_1 = calculate_y(i_0, b,
                  g_0 + Delta_g,
                  T, y_lim)
plot(0: T, y_1, title = "Aggregate Output",
     size=(600,400), xlabel = L"t",
     label = L"y_t")
```



Convergence and Uniqueness

Fixed Point Theory

- Fixed points, which will come about across a variety of places in economics
  - → Nash Equilibria, which requires fixed points of set-valued functions
  - → General Equilibrium
  - → Dynamic Programming - e.g., decision problems of macro agents
- Frequently in quantitative macro you will rewrite problems as fixed points in order to demonstrate uniqueness, convergence, and use fixed-point algorithms to solve

Convergence

- For $\boldsymbol{v}_{n+1} = \boldsymbol{f}(\boldsymbol{v}_n)$, take the limit for some $\boldsymbol{v}_0$,

$$
\begin{aligned}
v_1 &= f(v_0) \\
v_2 &= f(v_1) = f(f(v_0)) \\
&\dots \\
\lim_{n\to\infty} v_n &= f(f(\dots f((v_0)))) \stackrel{?}{\equiv} v^*
\end{aligned}
$$

  → Does this limit exist for all $\boldsymbol{v}_0$? (i.e, globally convergent)

  → Does it exist “local” to any $\boldsymbol{v}_0$? (i.e., locally convergent)

Uniqueness

- For $v_{n+1} = f(v_n)$, are there multiple fixed points?
  - → i.e., for some v_0 goes to v_1^* and for some v_0 goes to v_2^*
- Uniqueness should be interpreted in terms of economics
  - → Maybe non-uniqueness is interesting and leads to multiple equilibria (e.g., theories of growth where you can get stuck in a bad equilibria)
  - → Other times it says we wrote down the wrong model

Fixed Point Theorems

- A variety of fixed point theorems exist to show when solutions exist, and when solutions are unique
- For us, we can look at an especially simple one which provides necessary and sufficient conditions for convergence and uniqueness
  - → Banach's fixed-point theorem
  - → Useful because the proof is constructive (i.e., suggests algorithm)
  - → Gives us intuition on **contraction mappings**
- Lets stay in 1-dimensions $f : \mathbb{R} \to \mathbb{R}$, but can be generalized

Contraction Mappings

- A **contraction mapping** is a function f such that for some $0 < \beta < 1$ and all $x, y \in X$

$$|f(x) - f(y)| \leq \beta |x - y|$$

  → i.e., if I apply f to two points, the distance between the two points shrinks by a factor of β

Banach's Fixed Point Theorem

If f is a contraction mapping, then f has a **unique** fixed point x^*

- Moreover, for any x_0, the sequence $x_0, x_1, \ldots$ defined by $x_{n+1} = f(x_n)$ converges to x^*
- More generally: true on any on a complete metric space, but we won't need to generalize

Sketch of Proof

- The proof is constructive, and gives us a way to find the fixed point
- Start with $x_0 \in \mathbb{R}$ and define $x_{n+1} = f(x_n)$
- Then, for $n \geq 1$

$$\begin{aligned}|x_{n+1} - x_n| &= |f(x_n) - f(x_{n-1})| \leq \beta |x_n - x_{n-1}| = \beta |f(x_{n-1}) - f(x_{n-2})| \\ &\leq \beta^2 |x_{n-1} - x_{n-2}| \leq \cdots \leq \beta^n |x_1 - x_0|\end{aligned}$$

- Since $0 < \beta < 1$, the right hand side converges to zero as $n \to \infty$, independent of x_0
- Hence the $|x_{n+1} - x_n|$ goes to zero, so $x_n = x_{n+1} \to x^*$ as $n \to \infty$
  - → More subtle for fancier spaces X, but the same idea

Proving Contraction Mappings

- I won't ask you to do proofs in this class, but useful to see how you might do it
- Given this, a crucial tool is to be able to prove that a particular f is a contraction mapping
- Various ways to do this, and we will see connections to the gradient, $\nabla f(\cdot)$
- One useful theorem are called Blackwell's Sufficiency Conditions
- Sometimes it is easy to just apply the definition of **contraction mappings** directly

Example for Linear Functions

- Let $f(x) = a + bx$ for $a, b \in \mathbb{R}$
- Substitute into the the definition of **contraction mapping** directly

$$|f(x) - f(y)| = |a + bx - (a + by)| = |b||x - y| \leq \beta|x - y|$$

  → So f is a contraction mapping iff $\beta \equiv |b| < 1$

  → Consequently, f has a unique fixed point, $x^* = a + bx^*$

- The multidimensional generalization of this checks the maximum absolute eigenvalue