Stochastic Processes, Markov Chains, and Expectations

Graduate Quantitative Economics and Datascience

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Overview

Summary

- Here we build on the previous lecture on probability and distributions to introduce stochastic processes, Markov processes, and expectations/forecasts

We will introduce,

1. **Stochastic Processes** a sequence of events where the probability of the next event depends the past events
2. **Markov Processes** a stochastic process where the probability of the next event depends only on the current event

Extra Materials

- Intermediate QuantEcon Markov Chains
- QuantEcon AR1 Processes
- QuantEcon Markov Chains
- QuantEcon Simple Markov Chain Example

Packages

```
import matplotlib.pyplot as plt
import pandas as pd
import numpy as np
import scipy.stats
import seaborn as sns
from scipy.stats import rv_discrete
from numpy.linalg import matrix_power
```

Stochastic and Markov Processes

Discrete-time Stochastic Process

- A **stochastic process** is a sequence of random variables $\{X_t\}_{t=0}^{\infty}$ [1]
- Events in Ω are subtle to define because they contain nested information
  - → e.g. the realized random variable X_t depends on X_{t-1}, X_{t-2}, and changes the future random variables X_{t+1}, X_{t+2}, etc.
  - → Similarly, the probability of X_{t+1} is effected by the realized X_t and X_{t-1}
- Intuitively we can work with each $\{X_t\}_{t=0}^{\infty}$ and look at conditional distributions by considering independence, etc.

1. See formal definition **here**

Information Sets and Forecasts

- Expectations and conditional expectations give us notation for making forecasts while carefully defining information available
  - → More general, and not specific to stochastic processes or forecasts
  - → Might to “nowcast” or “smooth” to update your previous estimates
- To formalize
  1. Define **information set** as the known random variables
  2. Provide a random variable that is **forecast** using the information set
  3. Typically, provide a function of the random variable of interest and calculate the **conditional expectation** given the information set

Forecasts and Conditional Probability Distributions

- Take a stochastic process $\{X_t\}_{t=0}^{\infty}$
- Define the **information set** at t as $\mathcal{I}_t \equiv \{X_0, X_1, \dots, X_t\}$
- The **conditional probability** of X_{t+1} given the information set $\mathcal{I}_t$ is

$$\mathbb{P}(X_{t+1} \,|\, X_t, X_{t-1}, \dots X_0) \equiv \mathbb{P}(X_{t+1} \,|\, \mathcal{I}_t)$$

  → e.g. the probability of being unemployed, unemployed, or retired next period given the full workforce history

  → Useful in macroeconomics when you want to formalize expectations of the future, as well as econometrics when you want to update estimates given different amounts of observation

Forecasts and Conditional Expectations

- You may instead be interested in a function, $f(\cdot)$, of the random variable (e.g., financial payoffs, utility, losses in econometrics)
- Use the conditional probability of the forecasts for **conditional expectations**

$$\mathbb{E}[f(X_{t+1}) \,|\, X_t, X_{t-1}, \dots X_0] \equiv \mathbb{E}[f(X_{t+1}) \,|\, \mathcal{I}_t]$$

  → e.g. the expected utility of being unemployed next period given the history of unemployment; or the expected dividends in a portfolio next period given the history of dividends

- Standard properties of expectations hold conditioning on information sets,

  → $\mathbb{E}[A\, X_{t+1} + B\, Y_{t+1} \,|\, \mathcal{I}_t] = A\,\mathbb{E}[X_{t+1} \,|\, \mathcal{I}_t] + B\,\mathbb{E}[Y_{t+1} \,|\, \mathcal{I}_t]$

  → $\mathbb{E}[X_t \,|\, \mathcal{I}_t] = X_t$, i.e., not stochastic if the information set X_t

Easy Notation for Information Sets

- Information sets in stochastic processes are often just a sequence for the entire history. Hence the time, t, is often sufficient
- Given $\mathcal{I}_t \equiv \{X_0, X_1, \ldots, X_t\}$ for shorthand we can denote

$$
\begin{aligned}
\mathbb{E}[f(X_{t+1}) \,|\, X_t, X_{t-1}, \ldots X_0] &\equiv \mathbb{E}[f(X_{t+1}) \,|\, \mathcal{I}_t] \\
&\equiv \mathbb{E}_t[f(X_{t+1})]
\end{aligned}
$$

Law of Iterated Expectations for Stochastic Processes

- Recall that $\mathcal{I}_t \subset \mathcal{I}_{t+1}$ since X_{t+1} is known at $t+1$
- The **Law of Iterated Expectations** can be written as

$$\mathbb{E}\left[\mathbb{E}[X_{t+2} \mid X_{t+1}, X_t, X_{t-1}, \ldots] \mid X_t, X_{t-1}, \ldots\right] = \mathbb{E}[X_{t+2} \mid X_t, X_{t-1}, \ldots]$$
$$\mathbb{E}\left[\mathbb{E}[X_{t+2} \mid \mathcal{I}_{t+1}] \mid \mathcal{I}_t\right] = \mathbb{E}[X_{t+2} \mid \mathcal{I}_t]$$
$$\mathbb{E}_t[\mathbb{E}_{t+1}[X_{t+2}]] = \mathbb{E}_t[X_{t+2}]$$

- i.e. if I am forecasting my forecast, I can only use information available today

Markov Processes

- **(1st-Order) Markov Process**: a stochastic process where the conditional probability of the future is independent of the past given the present

$$\mathbb{P}(X_{t+1} \,|\, X_t, X_{t-1}, \ldots) = \mathbb{P}(X_{t+1} \,|\, X_t)$$

  → Or with information sets: $\mathbb{P}(X_{t+1} \,|\, \mathcal{I}_t) = \mathbb{P}(X_{t+1} \,|\, X_t)$

  → i.e., the present sufficiently summarizes the past for predicting the future

- **Conditional expectations** are are then

$$\mathbb{E}[f(X_{t+1}) \,|\, X_t, X_{t-1}, \ldots X_0] = \mathbb{E}[f(X_{t+1}) \,|\, X_t]$$

Martingales

- A stochastic process $\{X_t\}_{t=0}^{\infty}$ is a **martingale** if

$$\mathbb{E}[X_{t+1} \mid X_t, X_{t-1}, \dots, X_0] = X_t$$

- Not all martingales are Markov processes, but most of the ones you will encounter are. If Markov,

$$\mathbb{E}[X_{t+1} \mid X_t] = X_t, \quad \text{or} \quad \mathbb{E}_t[X_{t+1}] = X_t$$

See **here** for a more formal definition with the complete set of requirements

Random Walks

- Let $X_t \in \{-\infty, \dots, -1, 0, 1, \dots \infty\}$
- A simple two-state random walk can be written as the following transition

$$\mathbb{P}(X_{t+1} = X_t + 1 \,|\, X_t) = \mathbb{P}(X_{t+1} = X_t - 1 \,|\, X_t) = \frac{1}{2}$$

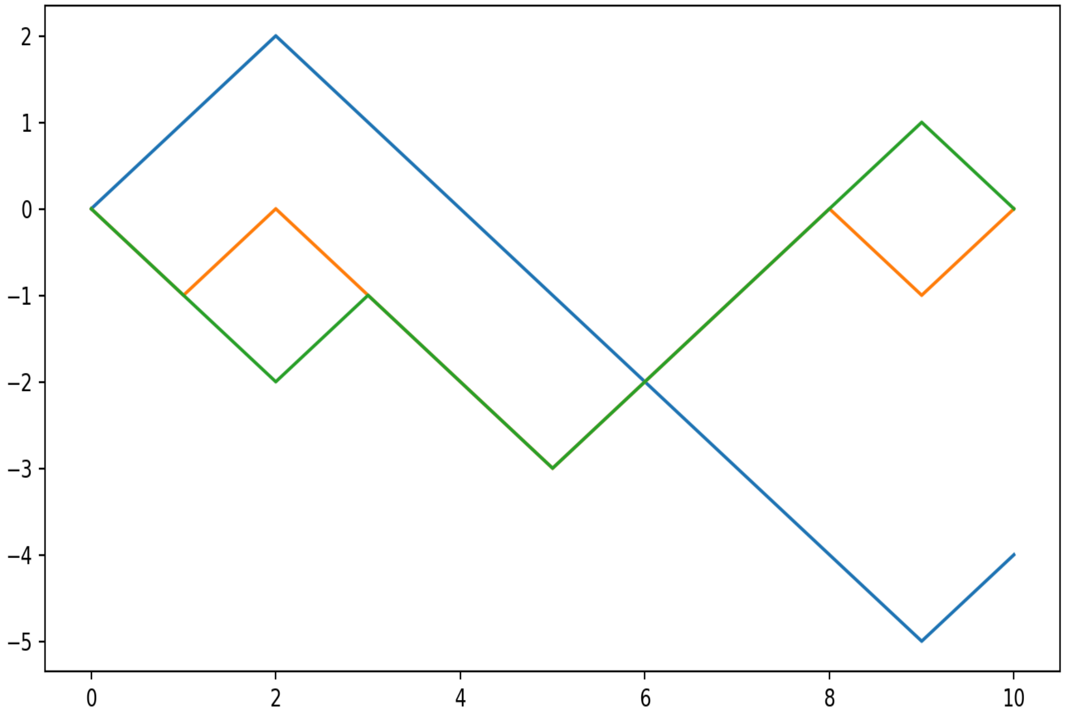
- Markov since X_t summarizes the past. Martingale?

$$\begin{aligned}\mathbb{E}(X_{t+1} \,|\, X_t) &= \mathbb{P}(X_{t+1} = X_t + 1 \,|\, X_t) \times (X_t + 1) \\ &\quad + \mathbb{P}(X_{t+1} = X_t - 1 \,|\, X_t) \times (X_t - 1) \\ &= \frac{1}{2}(X_t + 1) + \frac{1}{2}(X_t - 1) = X_t\end{aligned}$$

Implementation in Python

- Generic code to simulate a random walk with IID steps

```python
def simulate_walk(rv, X_0, T):
  X = np.zeros((X_0.shape[0], T+1))
  X[:, 0] = X_0
  for t in range(1, T+1):
    X[:, t] = X[:, t-1] \
              +rv.rvs(size=X_0.shape[0])
  return X
steps = np.array([-1, 1])
probs = np.array([0.5, 0.5])
rv = rv_discrete(values=(steps, probs))
X_0 = np.array([0.0, 0.0, 0.0])
X = simulate_walk(rv, X_0, 10)
plt.figure()
plt.plot(X.T)
```

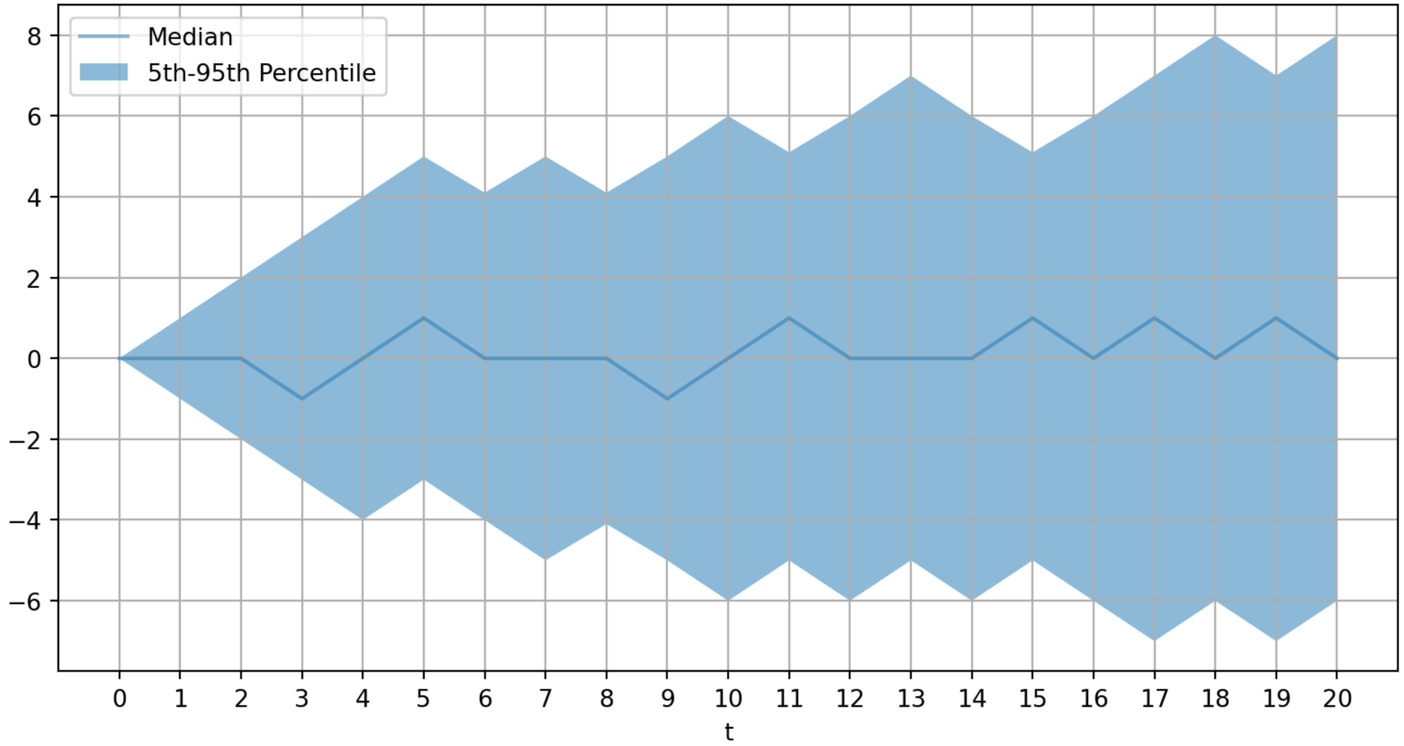


Visualizing the Distribution of Many Trajectories

- $\mathbb{E}_0[X_t] \to 0$ for finite t as $t \to \infty$
- But is there a limiting distribution of X_t as $X_t \to \infty$?

```python
num_trajectories, T = 100, 20
X = simulate_walk(rv, np.zeros(num_trajectories), T)
percentiles = np.percentile(X, [50, 5, 95], axis=0)
fig, ax = plt.subplots()
plt.plot(np.arange(T+1), percentiles[0,:], alpha=0.5, label='Median')
plt.fill_between(np.arange(T+1), percentiles[1,:], percentiles[2,:],
  alpha=0.5, label='5th-95th Percentile')
plt.xlabel('t')
ax.set_xticks(np.arange(T+1))
plt.legend()
plt.grid(True)
```

Visualizing the Distribution of Many Trajectories



AR(1) Processes

- An **auto-regressive process** of order 1, AR(1), is the Markov process

$$X_{t+1} = \rho X_t + \sigma \epsilon_{t+1}$$

  - → ρ is the **persistence** of the process, $\sigma \geq 0$ is the **volatility**
  - → ϵ_{t+1} is a random shock, we will assume $\mathcal{N}(0, 1)$
- Can show $X_{t+1} \,|\, X_t \sim \mathcal{N}(\rho X_t, \sigma^2)$ and hence

$$\mathbb{E}_t[X_{t+1}] = \rho X_t, \quad \mathbb{V}_t[X_{t+1}] = \sigma^2$$

For much more, see QuantEcon Lectures on AR(1)

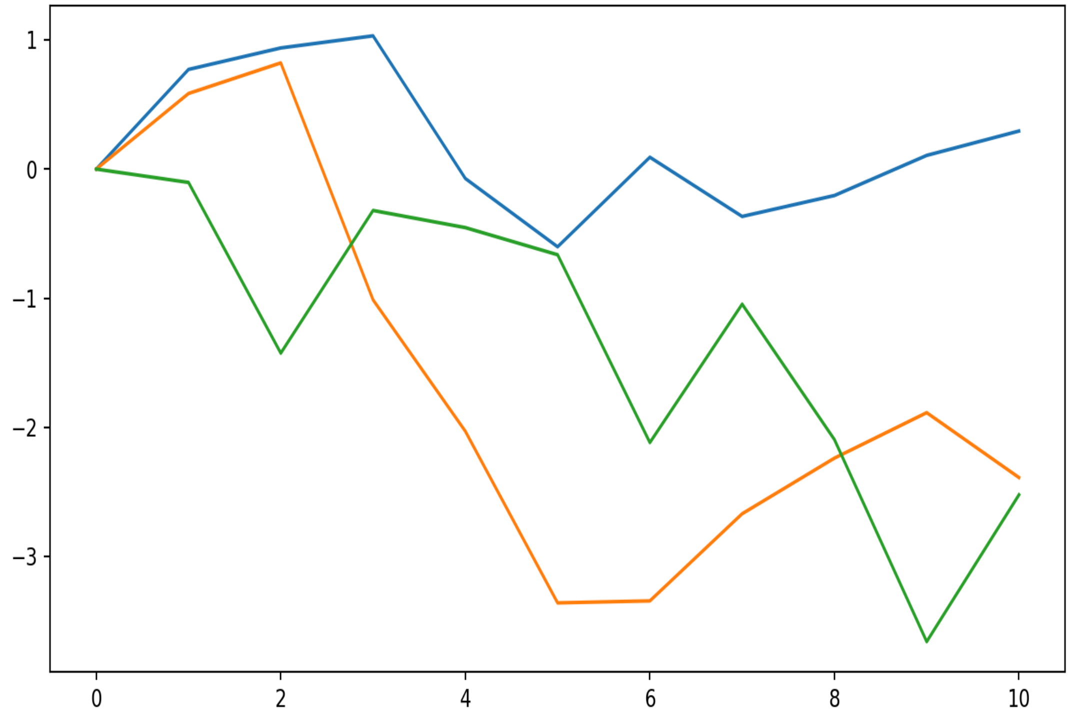
Stationarity and Unit Roots

- **Unit roots** are a special case of AR(1) processes where $\rho = 1$
- They are important in econometrics because they tell us if processes have permanent or transitory changes
  - → The econometrics of finding whether $\rho = 1$ are subtle and important
- Note that if $\rho = 1$ then this is a **martingale** since $\mathbb{E}_t[X_{t+1}] = X_t$
- These are an important example of a **non-stationary process**.
- Intuitively: stationary if X_t distribution has well-defined limit as $t \to \infty$
  - → Key requirements: $\lim_{t\to\infty} |\mathbb{E}[X_t]| < \infty$ and $\lim_{t\to\infty} \mathbb{V}(X_t) < \infty$

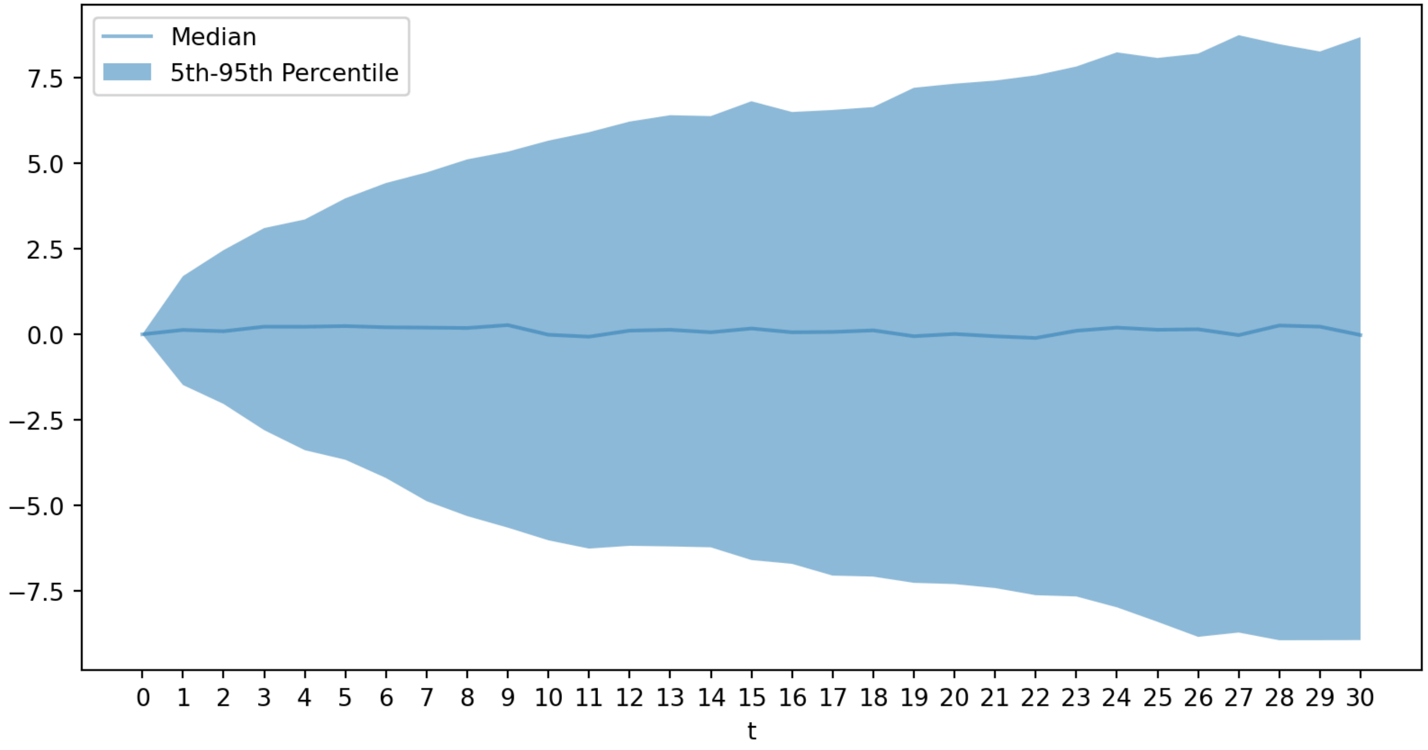
See here for a rigorous definitions and different types of stationarity and discussion of auto-covariance

Simulating Unit Root

```
X_0 = np.array([0.0, 0.0, 0.0])
rv_epsilon = scipy.stats.norm(loc=0, scale=1)
X = simulate_walk(rv_epsilon, X_0, 10)
plt.figure()
plt.plot(X.T)
```



Visualizing the Distribution of Many Trajectories



Martingales and Arbitrage in Finance

- **Random Walks** are a key model in finance
  - → e.g. stock prices, exchange rates, etc.
- Central to no-arbitrage pricing, after adjusting to interest rates/risk/etc.
  - → e.g. if you could predict the future price of a stock, you could make money by buying/selling today
  - → Martingales have no systematic drift which leads to a key source of arbitrage (especially with options/derivatives)
- Does this prediction hold up in the data? Generally yes, but depends on how you handle risk/etc.
  - → If it were systematically wrong then hedge funds and traders would be far richer than they are now

Information and Arbitrage

$$\mathbb{E}[X_{t+1} \mid \mathcal{I}_t] = X_t$$

- Given all of the information available, the best forecast of the future is the current price. Plenty of variables in $\mathcal{I}_t$ for individuals, including public prices
- Does this mean there is never arbitrage?
  - → No, just that it may be short-term because prices feed back into $\mathcal{I}_t$
  - → So some individuals make short term money given private information, but that information quickly becomes reflecting in other people's information sets (typically through prices)
  - → How, and how quickly markets aggregate information is a key question in financial economics

Markov Chains

Discrete-Time Markov Chains

- A **Markov Chain** is a Markov process with a finite number of states
  - → $X_t \in \{0, \ldots, N-1\}$ be a sequence of Markov random variables
  - → In discrete time it can be represented by a **transition matrix** P where

$$P_{ij} \equiv \mathbb{P}(X_{t+1} = j \,|\, X_t = i)$$

- We are counting from 0 to $N-1$ for coding convenience in Python. Names of discrete states are arbitrary!
  - → Count from 1 in R, Julia, Matlab, Fortran, instead

A continuous-time Markov Chain instead uses a **transition rate matrix** Λ where $\Lambda_{ij} = \lambda_{ij}$ is the rate of transitioning from state i to state j. All rows such to 0 rather than 1. Many properties have analogies, for example there is an eigenvalue of 0 rather than an eigenvalue of 1

Stochastic Matrices

- P is a stochastic matrix if
  - → $\sum_{j=0}^{N-1} P_{ij} = 1$ for all i, i.e. rows are conditional distributions
- **Key Property:**
  - → One (or more) eigenvalue of 1 with associated left-eigenvector π

$$\pi P = \pi$$

  - → Equivalently the right eigenvector with eigenvalue $= 1$

$$P^\top \pi^\top = \pi^\top$$

  - → Where we can normalize to $\sum_{n=0}^{N-1} \pi_i = 1$

Transitions and Conditional Distributions

- The P summarizes all transitions. Let X_t be the state at time t which in general is a probability distribution with pmf π_t
- Can show that the evolution of this distribution is given by

$$\pi_{t+1} = \pi_t \cdot P$$

- And hence given some X_t we can forecast the distribution of X_{t+j} with

$$X_{t+j} \mid X_t \sim \pi_t \cdot P^j$$

→ i.e., using the matrix power we discussed in previous lectures

Stationary Distribution

- Take some X_t initial condition, does this converge?

$$\lim_{j\to\infty} X_{t+j} \,|\, X_t = \lim_{j\to\infty} \pi_t \cdot P^j = \pi_\infty?$$

  - → Does it exist? Is it unique?
- How does it compare to fixed point below, i.e. does $\bar{\pi} = \pi_\infty$ for all X_t?

$$\bar{\pi} = \bar{\pi} \cdot P$$

  - → This is the eigenvector associated with the eigenvalue of 1 of $P^\top$
  - → Can prove there is always at least one. If more than one, multiplicity

The conditions for stationary distributions, uniqueness, etc. are covered **here**

Conditional Expectations

- Given the conditional probabilities, expectations are easy
- Now assign X_t as a random variable with values $x_1, \ldots x_N$ and pmf π_t
- Define $x \equiv \begin{bmatrix} x_0 & \ldots & x_{N-1} \end{bmatrix}$
- From definition of conditional expectations

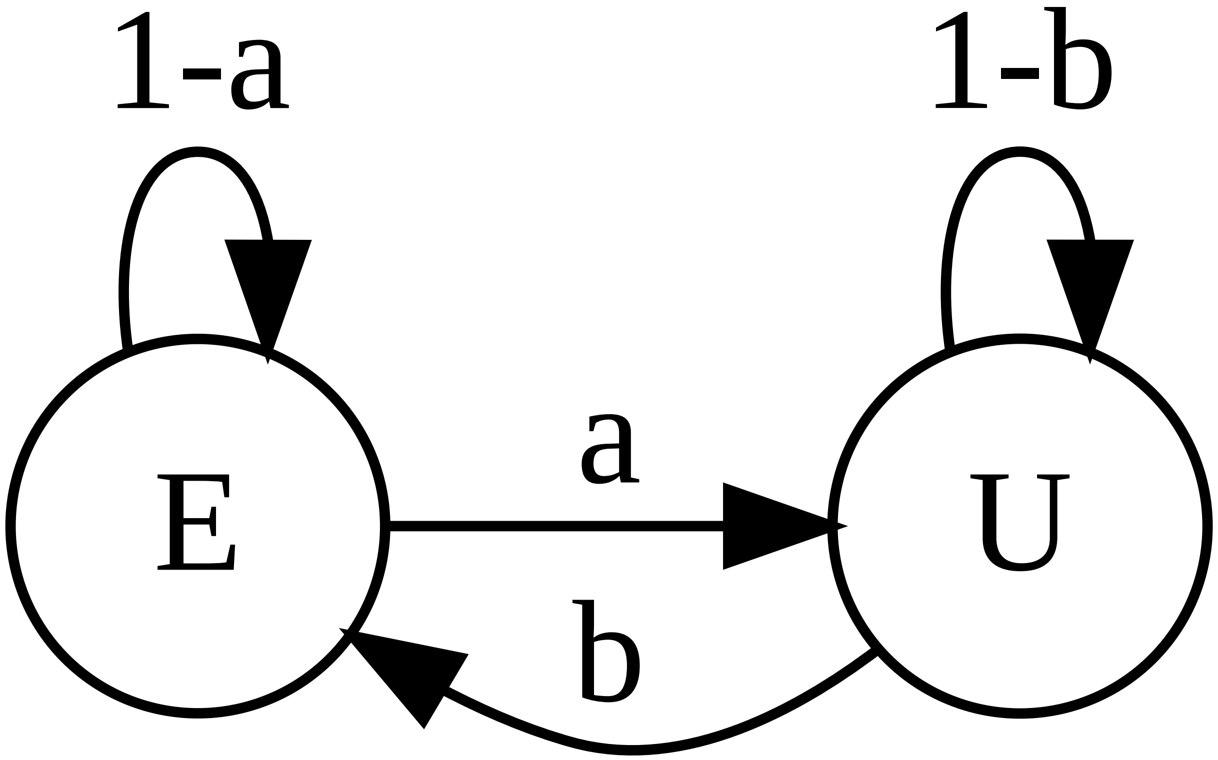
$$\mathbb{E}[X_{t+j} \,|\, X_t] = \sum_{i=0}^{N-1} x_i \pi_{t+j,i} = (\pi_t \cdot P^j) \cdot x$$

Example of Markov Chain: Employment Status

- Employment(E) in state $\mathbf{0}$, Unemployment(U) in state $\mathbf{1}$
- $\mathbb{P}(U \mid E) = a$ and $\mathbb{P}(E \mid E) = 1 - a$
- $\mathbb{P}(E \mid U) = b$ and $\mathbb{P}(U \mid U) = 1 - b$
- Transition matrix $P \equiv$

$$
\begin{array}{cc}
& \begin{array}{cc} \underbrace{X_{t+1}{=}E} & \underbrace{X_{t+1}{=}U} \end{array} \\
\begin{array}{c} X_t{=}E \} \\ X_t{=}U \} \end{array} & \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}
\end{array}
$$

Visualizing the Chain



Transitions and Probabilities

- Let $\pi_0 \equiv [1 \quad 0]^\top$, i.e. $\mathbb{P}(X_0 = E) = 1$
- The distribution of X_1 is then $\pi_1 = \pi_0 \cdot P$
  - → $\mathbb{P}(X_1 = E \,|\, X_0 = E) = \pi_{11}$ (first element)
  - → Can use to forecast probability of employment j periods in future
- Can also use our conditional expectations to calculate expected income
  - → Define income in E state as $100,000$ and $20,000$ in the U
  - → $x \equiv [100,000 \quad 20,000]^\top$

$$\mathbb{E}[X_{t+j} \,|\, X_t = E] = ([1 \quad 0] \cdot P^j) \cdot x$$

Coding Markov Chain in Python

- We can make simulation easier if turn rows into conditional distributions
- Count states from 0 to make coding easier, i.e. $E = 0$ and $U = 1$

```python
a, b = 0.05, 0.1
P = np.array([[1-a, a],  # P(X | E)
              [b, 1-b]]) # P(X | U)
N = P.shape[0]
P_rv = [rv_discrete(values=(np.arange(0,N),
                    P[i,:])) for i in range(N)]
X_0 = 0 # i.e. E
X_1 = P_rv[X_0].rvs() # draw index | X_0
print(f"X_0 = {X_0}, X_1 = {X_1}")
T = 10
X = np.zeros(T+1, dtype=int)
X[0] = X_0
for t in range(T):
  X[t+1] = P_rv[X[t]].rvs() # draw given X_t
print(f"X_t indices =\n {X}")
```

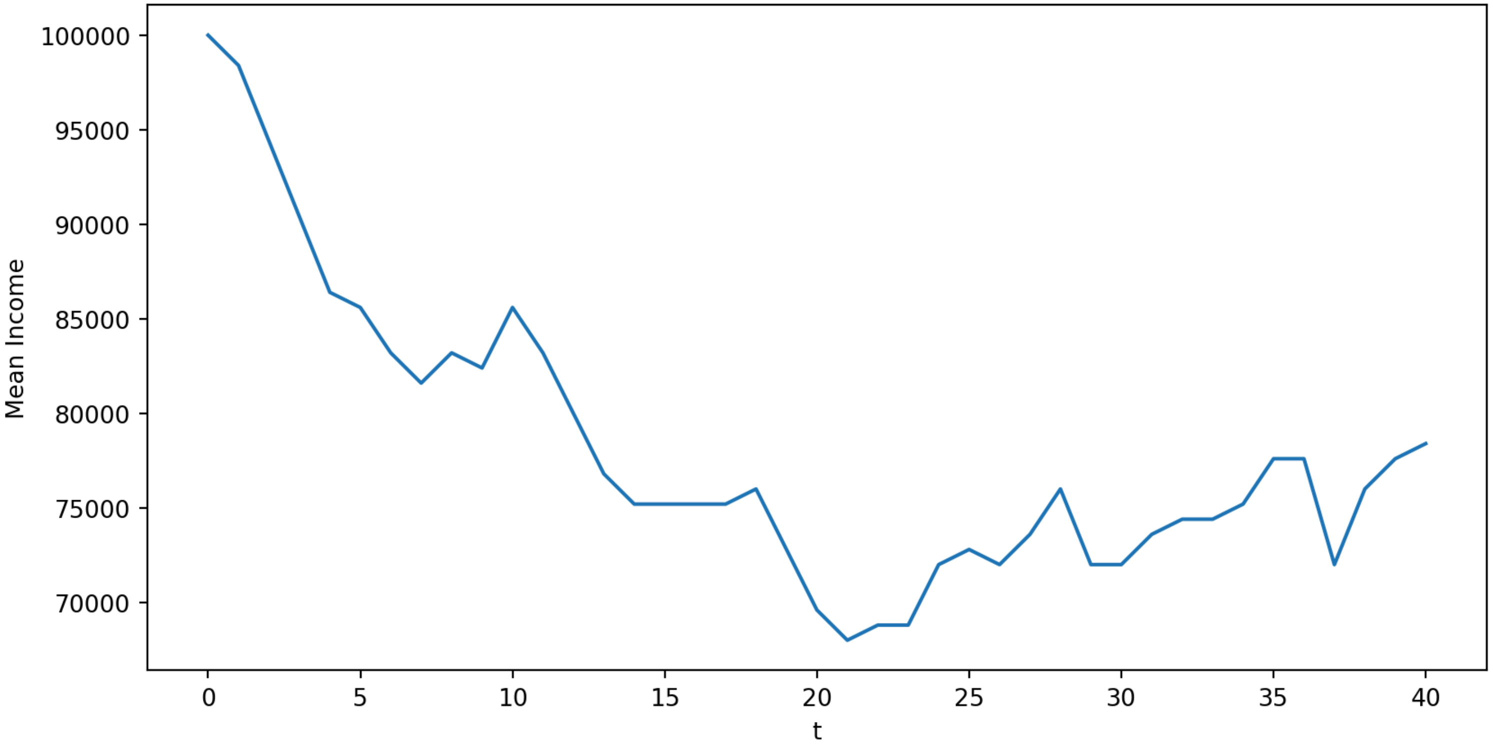
```
X_0 = 0, X_1 = 0
X_t indices =
 [0 0 0 0 0 0 0 0 0 0 0]
```

Simulating Many Trajectories

```python
def simulate_markov_chain(P, X_0, T):
  N = P.shape[0]
  num_chains = X_0.shape[0]
  P_rv = [rv_discrete(values=(np.arange(0,N),
                    P[i,:])) for i in range(N)]
  X = np.zeros((num_chains, T+1), dtype=int)
  X[:,0] = X_0
  for t in range(T):
      for n in range(num_chains):
          X[n, t+1] = P_rv[X[n, t]].rvs()
  return X
X_0 = np.zeros(100, dtype=int) # 100 people start employed
T = 40
X = simulate_markov_chain(P, X_0, T)
# Map indices to RV values
values = np.array([100000.00, 20000.00]) # map state to value
X_values = values[X] # just indexes by the X

# Plot means
```

Simulating Many Trajectories



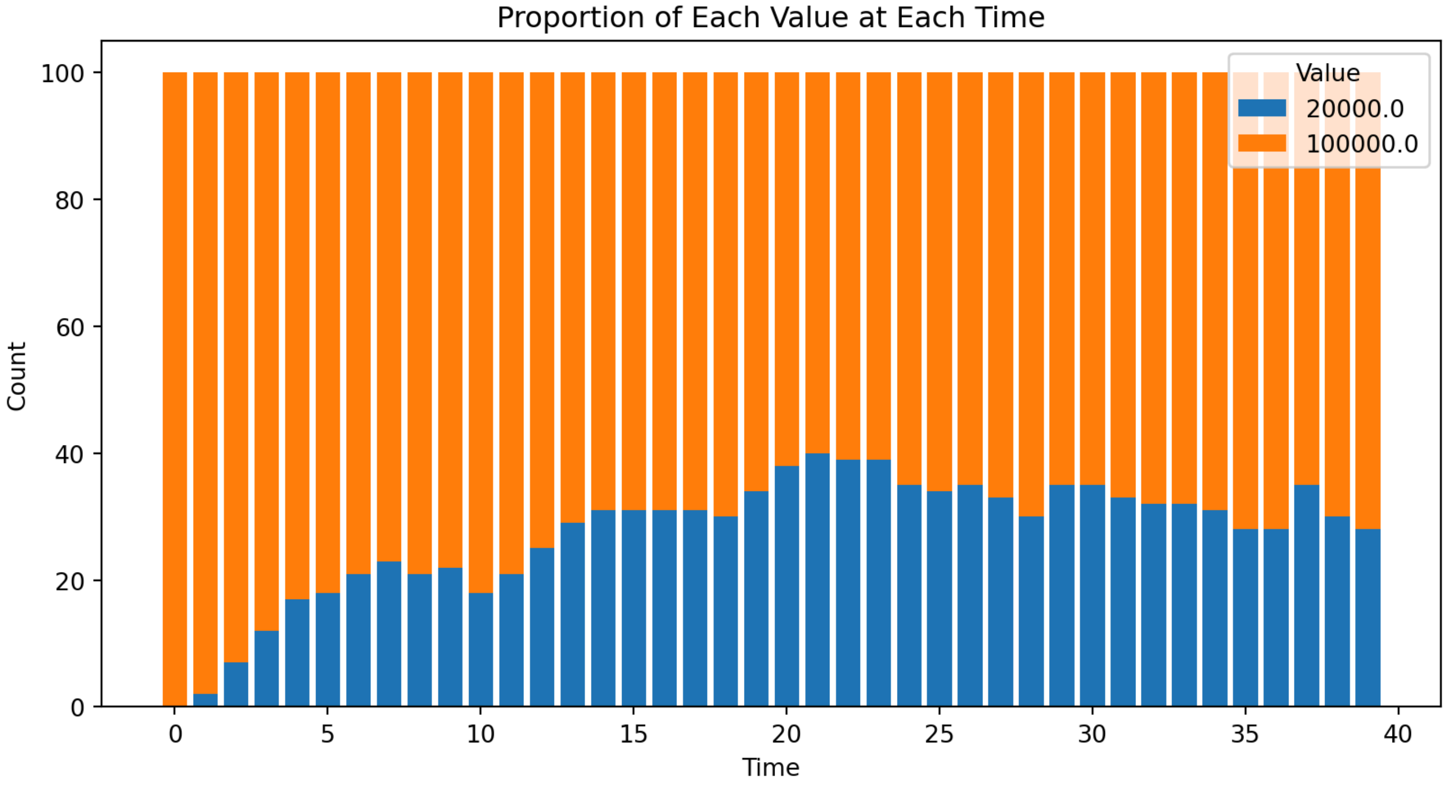
Visualizing the Distribution of Many Trajectories

```python
# Count the occurrences of each unique value at each time step
unique_values = np.unique(X_values)
counts = np.array([[np.sum(X_values[:, t] == val) for val in unique_values] for t in range(T)])

# Create the stacked bar chart
fig, ax = plt.subplots()
bottoms = np.zeros(T)
for i, val in enumerate(unique_values):
    ax.bar(range(T), counts[:, i], bottom=bottoms, label=str(val))
    bottoms += counts[:, i]

# Labels and title
ax.set_xlabel('Time')
ax.set_ylabel('Count')
ax.set_title('Proportion of Each Value at Each Time')
ax.legend(title='Value')
plt.show()
```

Visualizing the Distribution of Many Trajectories



Stationary Distribution

- Recall different ways to think about steady states
  - → Left-eigenvector: $\bar{\pi} = \bar{\pi} P$
  - → Limiting distribution: $\lim_{T \to \infty} \pi_0 P^T$
- Can show that the stationary distribution is $\bar{\pi} = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$

```
eigvals, eigvecs = np.linalg.eig(P.T)
pi_bar = eigvecs[:, np.isclose(eigvals, 1)].flatten().r
pi_bar = pi_bar / pi_bar.sum()
pi_0 = np.array([1.0, 0.0])
pi_inf = pi_0 @ matrix_power(P, 100)
print(f"pi_bar = {pi_bar}")
print(f"pi_inf = {pi_inf}")
```

```
pi_bar = [0.66666667 0.33333333]
pi_inf = [0.6666667 0.3333333]
```

Expected Income

- Recall that $\mathbb{E}[X_{t+j} \,|\, X_t = E] = ([1 \quad 0] \cdot P^j) \cdot x$

```python
def forecast_distributions(P, pi_0, T):
    N = P.shape[0]
    pi = np.zeros((T+1, N))
    pi[0, :] = pi_0
    for t in range(T):
        pi[t+1, :] = pi[t, :] @ P
    return pi
x = np.array([100000.00, 20000.00])
pi_0 = np.array([1.0, 0.0])
T = 20
pi = forecast_distributions(P, pi_0, T)
E_X_t = np.dot(pi, x)
E_X_bar = pi_bar @ x
plt.plot(np.arange(0, T+1), E_X_t)
plt.axhline(E_X_bar, color='r',
  linestyle='--')
plt.show()
```