Applications of Linear Algebra and Eigenvalues

Graduate Quantitative Economics and Datascience

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Overview

Motivation and Materials

- In this lecture, we will cover some applications of the tools we developed in the previous lecture
- The goal is to build some useful tools to sharpen your intuition on linear algebra and eigenvalues/eigenvectors, and practice some basic coding

Extra Materials

- Material related to: QuantEcon Python, QuantEcon Data Science, Intro Quantitative Economics with Python
- **Self-study and Optional Materials:**
  - → Geometric Series and Present Values
  - → Portfolio example
  - → Unemployment Dynamics example
  - → Supply and Demand
  - → More on Competitive Equilibrium

Packages

```
import numpy as np
import matplotlib.pyplot as plt
import scipy
from numpy.linalg import cond, matrix_rank, norm
from scipy.linalg import inv, solve, det, eig, lu, eigvals
from scipy.linalg import solve_triangular, eigvalsh, cholesky
```

Difference Equations

Linear Difference Equations as Iterative Maps

- Consider $A : \mathbb{R}^N \to \mathbb{R}^N$ as the linear map for the state $x_t \in \mathbb{R}^N$
- An example of a linear difference equation is

$$x_{t+1} = Ax_t$$

where

$$A \equiv \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.8 \end{bmatrix}$$

```python
A = np.array([[0.9, 0.1], [0.5, 0.8]])
x_0 = np.array([1, 1])
x_1 = A @ x_0
print(f"x_1 = {x_1}, x_2 = {A @ x_1}")
```

```
x_1 = [1.  1.3], x_2 = [1.03 1.54]
```

Iterating with $\rho(A) > 1$

Iterate $x_{t+1} = Ax_t$ from x_0 for $t = 100$

```
x_0 = np.array([1, 1])
t = 200
x_t = np.linalg.matrix_power(A, t) @ x_0
rho_A = np.max(np.abs(eigvals(A)))
print(f"rho(A) = {rho_A}")
print(f"x_{t} = {x_t}")
```

```
rho(A) = 1.079128784747792
x_200 = [3406689.32410673 6102361.18640516]
```

- Diverges to $x_\infty = \begin{bmatrix}\infty & \infty\end{bmatrix}^T$
- $\rho = 1 + 0.079$ says in the worst case (i.e., $x_t \propto$ the eigenvector associated with $\lambda = 1.079$ eigenvalue), expands by 7.9% on each iteration

Iterating with $\rho(A) < 1$

```python
A = np.array([[0.6, 0.1], [0.5, 0.8]])
x_t = np.linalg.matrix_power(A, t) @ x_0
rho_A = np.max(np.abs(eigvals(A)))
print(f"rho(A) = {rho_A}")
print(f"x_{t} = {x_t}")
```

```
rho(A) = 0.9449489742783178
x_200 = [6.03450418e-06 2.08159603e-05]
```

- Converges to $x_\infty = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$

Iterating with $\rho(A) = 1$

- To make a matrix that has $\rho(A) = 1$ reverse eigendecomposition!
- Leave previous eigenvectors in Q, change Λ to force $\rho(A)$ directly

```
Q = np.array([[-0.85065081, -0.52573111], [0.52573111, -0.85065081]])
print(f"check orthogonal: dot(x_1,x_2) approx 0: {np.dot(Q[:,0], Q[:,1])}")
Lambda = [1.0, 0.8]  # choosing eigenvalue so max_n|lambda_n| = 1
A = Q @ np.diag(Lambda) @ inv(Q)
print(f"rho(A) = {np.max(np.abs(eigvals(A)))}")
print(f"x_{t} = {np.linalg.matrix_power(A, t) @ x_0}")
```

```
check orthogonal: dot(x_1,x_2) approx 0: 0.0
rho(A) = 1.0
x_200 = [ 0.27639321 -0.17082039]
```

Unemployment Dynamics

Dynamics of Employment without Population Growth

- Consider an economy where in a given year $\alpha = 5\%$ of employed workers lose job and $\phi = 10\%$ of unemployed workers find a job
- We start with $E_0 = 900,000$ employed workers, $U_0 = 100,000$ unemployed workers, and no birth or death. Dynamics for the year:

$$
\begin{aligned}
E_{t+1} &= (1-\alpha)E_t + \phi U_t \\
U_{t+1} &= \alpha E_t + (1-\phi)U_t
\end{aligned}
$$

Write as Linear System

- Use matrices and vectors to write as a linear system

$$\underbrace{\begin{bmatrix} E_{t+1} \\ U_{t+1} \end{bmatrix}}_{X_{t+1}} = \underbrace{\begin{bmatrix} 1-\alpha & \phi \\ \alpha & 1-\phi \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} E_t \\ U_t \end{bmatrix}}_{X_t}$$

Simulating

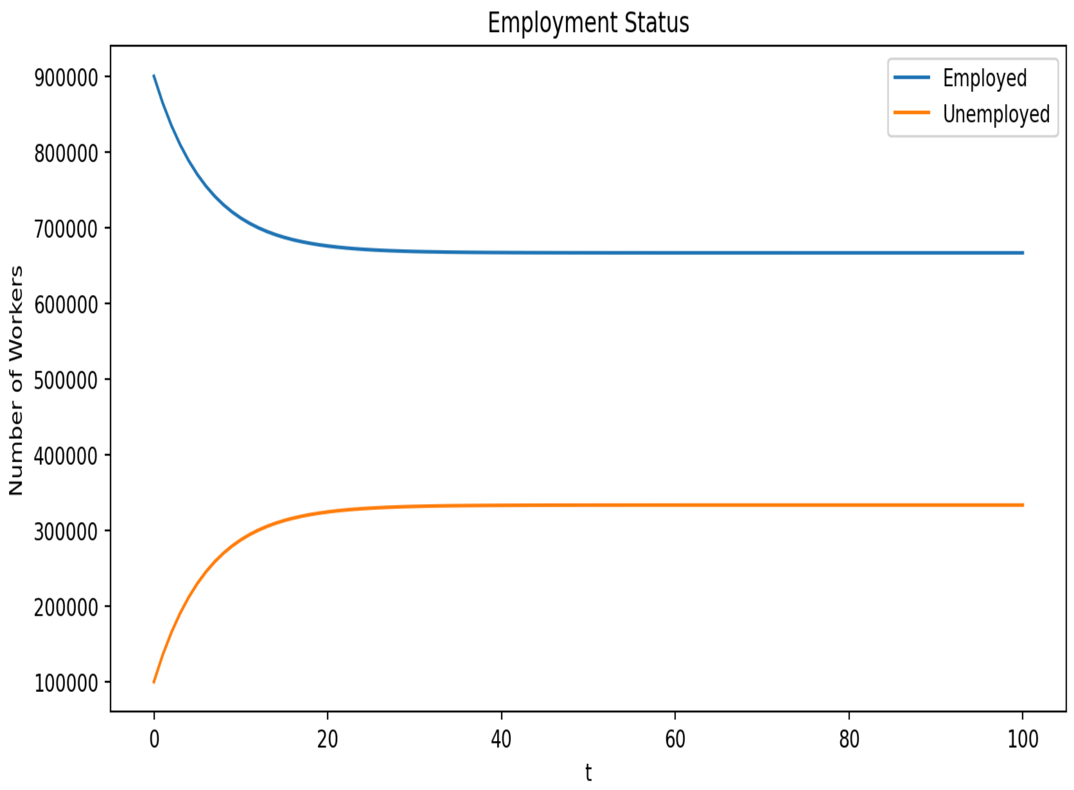
Simulate by iterating $X_{t+1} = AX_t$ from X_0 until $T = 100$

```python
def simulate(A, X_0, T):
    X = np.zeros((2, T+1))
    X[:,0] = X_0
    for t in range(T):
        X[:,t+1] = A @ X[:,t]
    return X
X_0 = np.array([900000, 100000])
A = np.array([[0.95, 0.1], [0.05, 0.9]])
T = 100
X = simulate(A, X_0, T)
print(f"X_{T} = {X[:,T]}")
```

```
X_100 = [666666.6870779  333333.31292209]
```

Dynamics of Unemployment

```
fig, ax = plt.subplots()
ax.plot(range(T+1), X.T,
  label=["Employed", "Unemployed"])
ax.set(xlabel="t",
  ylabel="Number of Workers",
  title="Employment Status")
ax.legend()
plt.show()
```



Convergence to a Longrun Distribution

- Find X_∞ by iterating $X_{t+1} = AX_t$ many times from a X_0?
  - → Check if it has converged with $X_\infty \approx AX_\infty$
  - → Is X_∞ the same from any X_0? Will discuss "ergodicity" later
- Alternatively, note that this expression is the same as

$$1 \times \bar{X} = A\bar{X}$$

  - → i.e, a $\lambda = 1$ where $\bar{X}$ is the corresponding eigenvector of A
  - → Is $\lambda = 1$ always an eigenvalue? (yes if all $\sum_{n=1}^{N} A_{ni} = 1$ for all i)
  - → Does $\bar{X} = X_\infty$? For any X_0?
  - → Multiple eigenvalues with $\lambda = 1 \implies$ multiple $\bar{X}$

Using the First Eigenvector for the Steady State

```python
Lambda, Q = eig(A)
print(f"real eigenvalues = {np.real(Lambda)}")
print(f"eigenvectors in columns of =\n{Q}")
print(f"first eigenvalue = 1? \
{np.isclose(Lambda[0], 1.0)}")
X_bar = Q[:,0] / np.sum(Q[:,0]) * np.sum(X_0)
print(f"X_bar = {X_bar}\nX_{T} = {X[:,T]}")
```

```
real eigenvalues = [1.   0.85]
eigenvectors in columns of =
[[ 0.89442719 -0.70710678]
 [ 0.4472136   0.70710678]]
first eigenvalue = 1? True
X_bar = [666666.66666667 333333.33333333]
X_100 = [666666.6870779  333333.31292209]
```

Using the Second Eigenvalue for the Convergence Speed

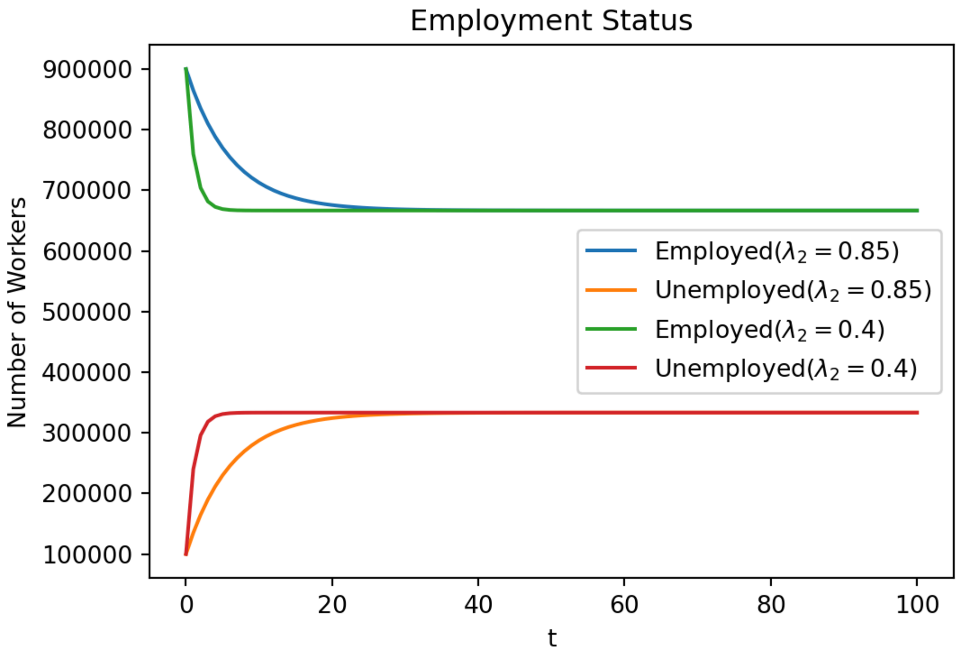
- The second largest ($\lambda_2 < 1$) provides information on the speed of convergence
  - → 0 is instantaneous convergence here
  - → 1 is no convergence here
- We will create a new matrix with the same steady state, different speed
  - → To do this, build a new matrix with the same eigenvectors (in particular the same eigenvector associated with the $\lambda = 1$ eigenvalue)
  - → But we will replace the eigenvalues $[1.0 \quad 0.85]$ with $[1.0 \quad 0.5]$
  - → Then we will reconstruct A matrix and simulate again
- Intuitively we will see the that the resulting A_{fast} implies α and ϕ which are larger by the same proportion

Simulating with Different Eigenvalues

```python
Lambda_fast = np.array([1.0, 0.4])
A_fast = Q @ np.diag(Lambda_fast) @ inv(Q) # same eigenvectors
print("A_fast =\n", A_fast)
print(f"alpha_fast/alpha = {A_fast[1,0]/A[1,0]:.2g}, \
phi_fast/phi = {A_fast[0,1]/A[0,1]:.2g}")
X_fast = simulate(A_fast, X_0, T)
print(f"X_{T} = {X_fast[:,T]}")
```

```
A_fast =
 [[0.8 0.4]
 [0.2 0.6]]
alpha_fast/alpha = 4, phi_fast/phi = 4
X_100 = [666666.66666667 333333.33333333]
```

Convergence Dynamics of Unemployment



Present Discounted Values

Geometric Series

- Assume dividends follow $y_{t+1} = G y_t$ for $t = 0, 1, \ldots$ and y_0 is given
- $G > 0$, dividends are discounted at factor $\beta > 1$ then $p_t = \sum_{s=0}^{\infty} \beta^s y_{t+s} = \frac{y_t}{1-\beta G}$
- More generally if $x_{t+1} = A x_t$, $x_t \in \mathbb{R}^N$, $y_t = G x_t$ and $A \in \mathbb{R}^{N \times N}$, then

$$
\begin{aligned}
p_t &= y_t + \beta y_{t+1} + \beta^2 y_{t+2} + \ldots = G x_t + \beta G A x_t + \beta^2 G A A x_t + \ldots \\
&= \sum_{s=0}^{\infty} \beta^s G A^s y_t \\
&= G(I - \beta A)^{-1} x_t \quad , \text{ if } \rho(A) < 1/\beta
\end{aligned}
$$

- where $\rho(A)$ is the spectral radius

Discounting and the Spectral Radius

- Intuitively, the spectral radius of A, the maximum scaling, must be less than discounting
- Intuition from univariate:
  - → If $G \in \mathbb{R}^{1\times 1}$ then $\mathbf{eig}(G) = G$, so must have $|\beta G| < 1$

PDV Example

Here is an example with $1 < \rho(A) < 1/\beta$. Try with different A

```
beta = 0.9
A = np.array([[0.85, 0.1], [0.2, 0.9]])
G = np.array([[1.0, 1.0]]) # row vector
x_0 = np.array([1.0, 1.0])
p_t = G @ solve(np.eye(2) - beta * A, x_0)
#p_t = G @ inv(np.eye(2) - beta * A) @ x_0 # alternative
rho_A = np.max(np.abs(np.real(eigvals(A))))
print(f"p_t = {p_t[0]:.4g}, spectral radius = {rho_A:.4g}, 1/beta = {1/beta:.4g}")
```

```
p_t = 24.43, spectral radius = 1.019, 1/beta = 1.111
```

(Optional) Matrix Conditioning and Stability

Matrix Conditioning

- Poorly conditioned matrices can lead to inaccurate or wrong solutions
- Tends to happen when matrices are close to singular or when they have very different scales - so there will be times when you need to rescale your problems

```
eps = 1e-7
A = np.array([[1, 1], [1 + eps, 1]])
print(f"A =\n{A}")
print(f"A^{-1} =\n{inv(A)}")
```

```
A =
[[1.         1.        ]
 [1.0000001 1.        ]]
A^-1 =
[[-9999999.99336215  9999999.99336215]
 [10000000.99336215 -9999999.99336215]]
```

Condition Numbers of Matrices

- $\det(A) \approx 0$ may say it is "almost" singular, but it is not scale-invariant
- $\mathbf{cond}(A) \equiv ||A|| \cdot ||A^{-1}||$ where $||\cdot||$ is the matrix norm - expensive to calculate in practice. Connected to eigenvalues $\mathbf{cond}(A) = |\frac{\lambda_{max}}{\lambda_{min}}|$
- Scale free measure of numerical issues for a variety of matrix operations
- Intuition: if $\mathbf{cond}(A) = K$, then $b \to b + \nabla b$ change in b amplifies to a $x \to x + K\nabla b$ error when solving $Ax = b$.
- See Matlab Docs on inv for example, where `inv` is a bad idea due to poor conditioning

```python
print(f"condition(I) = {cond(np.eye(2))}")
print(f"condition(A) = {cond(A)}, condition(A^(-1)) = {cond(inv(A))}")
```

```
condition(I) = 1.0
condition(A) = 40000001.962777555, condition(A^(-1)) = 40000002.02779216
```

Example with Interpolation

- Consider fitting data $\boldsymbol{x} \in \mathbb{R}^{N+1}$ and $\boldsymbol{y} \in \mathbb{R}^{N+1}$ with an N-degree polynomial
- That is, find $\boldsymbol{c} \in \mathbb{R}^{N+1}$ such that

$$
\begin{aligned}
c_0 + c_1 x_1 + c_2 x_1^2 + \ldots + c_N x_1^N &= y_1 \\
\ldots &= \ldots \\
c_0 + c_1 x_N + c_2 x_N^2 + \ldots + c_N x_N^N &= y_N
\end{aligned}
$$

- Which we can then use as $P(x) = \sum_{n=0}^{N} c_n x^n$ to interpolate between the points

Writing as a Linear System

- Define a matrix of all of the powers of the $\boldsymbol{x}$ values

$$A \equiv \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^N \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_N & x_N^2 & \dots & x_N^N \end{bmatrix}$$

- Then solve for $\boldsymbol{c}$ as the solution (where $\boldsymbol{A}$ is invertible if $\boldsymbol{x_n}$ are unique)

$$Ac = y$$

Solving an Example

- Let's look at the numerical error here from the interpolation using the inf-norm, i.e., $||x||_\infty = \max_n |x_n|$

```
N = 5
x = np.linspace(0.0, 10.0, N + 1)
y = np.exp(x)  # example function to interpolate
A = np.array([[x_i**n for n in range(N + 1)] for x_i in x])  # or np.vander
c = solve(A, y)
c_inv = inv(A) @ y
print(f"error = {norm(A @ c - y, np.inf)}, \
error using inv(A) = {norm(A @ c_inv - y, np.inf)}")
print(f"cond(A) = {cond(A)}")
```

```
error = 1.574562702444382e-11, error using inv(A) = 1.1932570487260818e-09
cond(A) = 564652.3214000753
```

Things Getting Poorly Conditioned Quickly

```python
N = 10
x = np.linspace(0.0, 10.0, N + 1)
y = np.exp(x)  # example function to interpolate
A = np.array([[x_i**n for n in range(N + 1)] for x_i in x])  # or np.vander
c = solve(A, y)
c_inv = inv(A) @ y # Solving with inv(A) instead of solve(A, y)
print(f"error = {norm(A @ c - y, np.inf)}, \
error using inv(A) = {norm(A @ c_inv - y, np.inf)}")
print(f"cond(A) = {cond(A)}")
```

```
error = 5.334186425898224e-10, error using inv(A) = 6.22717197984457e-06
cond(A) = 4462824600195.809
```

Matrix Inverses Fail Completely for $N = 20$

```
N = 20
x = np.linspace(0.0, 10.0, N + 1)
y = np.exp(x)  # example function to interpolate
A = np.array([[x_i**n for n in range(N + 1)] for x_i in x])  # or np.vander
c = solve(A, y)
c_inv = inv(A) @ y # Solving with inv(A) instead of solve(A, y)
print(f"error = {norm(A @ c - y, np.inf)}, \
error using inv(A) = {norm(A @ c_inv - y, np.inf)}")
print(f"cond(A) = {cond(A):.4g}")
```

```
error = 8.36735125631094e-10, error using inv(A) = 1419.6725472353137
cond(A) = 2.938e+24
```

Moral of this Story

- Use `solve`, which is faster and can often solve ill-conditioned problems. Rarely use `inv`, and only when you know the problem is well-conditioned
- Check conditioning of matrices when doing numerical work as an occasional diagnostic, as it is a good indicator of potential problems and collinearity
- For approximation, never use a monomial basis for polynomials
  - → Prefer polynomials like Chebyshev, which are designed to be as orthogonal as possible

```
N = 40
x = np.linspace(-1, 1, N+1)  # Or any other range of x values
A = np.array([[np.polynomial.Chebyshev.basis(n)(x_i) for n in range(N+1)] for x_i in x])
A_monomial = np.array([[x_i**n for n in range(N + 1)] for x_i in x])  # or np.vander
print(f"cond(A) = {cond(A):.4g}, cond(A_monomial) = {cond(A_monomial):.4g}")
```

```
cond(A) = 3.64e+09, cond(A_monomial) = 5.311e+17
```