Linear and Nonlinear Dynamics

Graduate Quantitative Economics and Datascience

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Overview

Motivation and Materials

- In this lecture, we will apply some of our tools to non-linear equations, which come up in macroeconomics, industrial organization, and econometrics
- The primary example is a simple version of the growth models
- We will introduce the idea of a fixed point, which has many applications across fields of economics
- A special emphasis will be placed on analyzing stability - which connects to the eigenvalues of the dynamical system

Extra Materials

- Solow-Swan Model (skip 20.3)
- Dynamics and Stability in One Dimension
- Nonlinear Dynamics and Stability
- Review taylor series, just to first order
- More on the Solow Model and Python

Packages

```
import matplotlib.pyplot as plt
import numpy as np
from numpy.linalg import norm
from scipy.linalg import inv, solve, det, eig, lu, eigvals
```

Fixed Points

Fixed Points of a Map

Fixed Point

Let $f : S \to S$ where we will assume $S \subseteq \mathbb{R}^N$. Then a fixed point $x^* \in S$ of f is one where

$$x^* = f(x^*)$$

Fixed points **may not exist**, or could have **multiplicity**

Fixed Points for Linear Functions

- We have already done this for linear functions.
- Let $f(x) = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix} x$
- Then we know that $x^* = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ is a fixed point
- Are there non-trivial others?
  - → Could check eigevectors as we did before, $\lambda \times x = Ax$
  - → If there is an (λ, x) pair with $\lambda = 1$ it is a fixed point

```
A = np.array([[0.8, 0.2], [0.2, 0.8]])
eigvals, eigvecs = eig(A)
print(f"lambda_1={eigvals[0]}, ||x* - A x*||={norm(A @ eigvecs[:,0] - eigvecs[:,0])}")
```

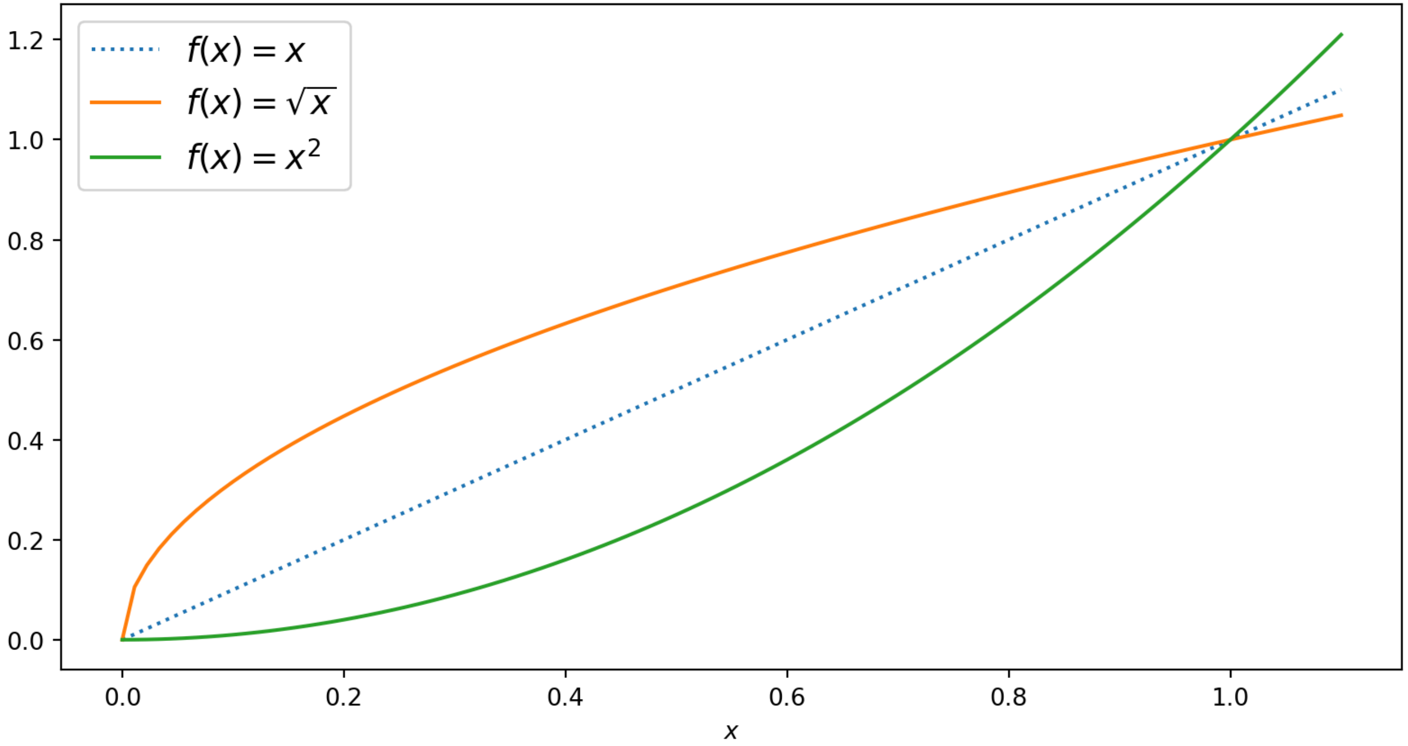
```
lambda_1=(1+0j), ||x* - A x*||=1.1102230246251565e-16
```

Fixed Points for Nonlinear Functions

- Consider $f(x) = \sqrt{x}$ and $f(x) = x^2$ for $x \geq 0$
- Trivially $x^* = 0$ is a fixed point of both, but what about others?
- Plot the 45-degree line to see if they cross! Seems $x^* = 1$ as well?
  - → As we will discuss, though. The shape at $x^* = 1$ and $x^* = 0$ is very different
  - → Think about what happens if we "perturb" slightly away from that point?

Plot Against 45 degree line

- Consider $f(x) = \sqrt{x}$ and $f(x) = x^2$ for $x \geq 0$

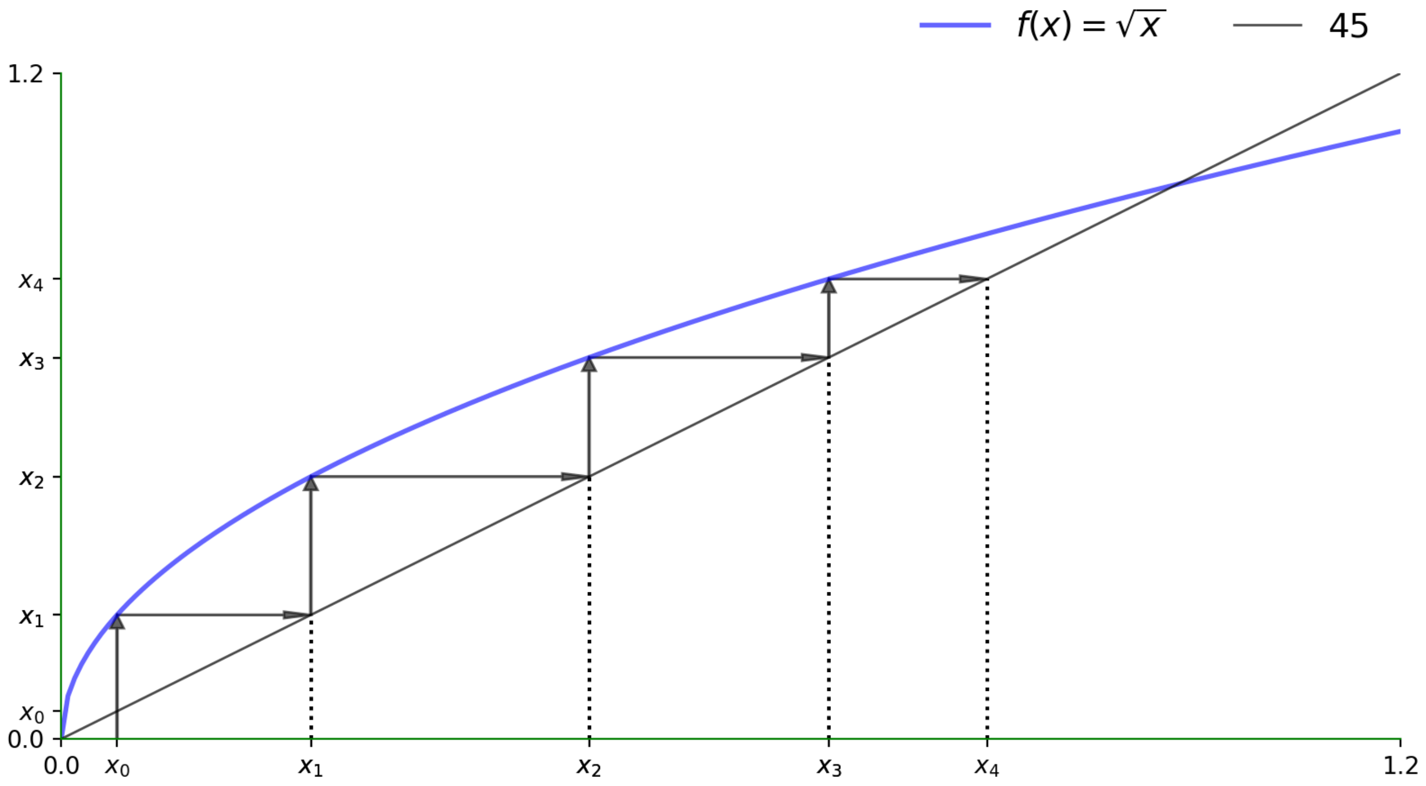


Interpreting Iterations with the 45 degree line

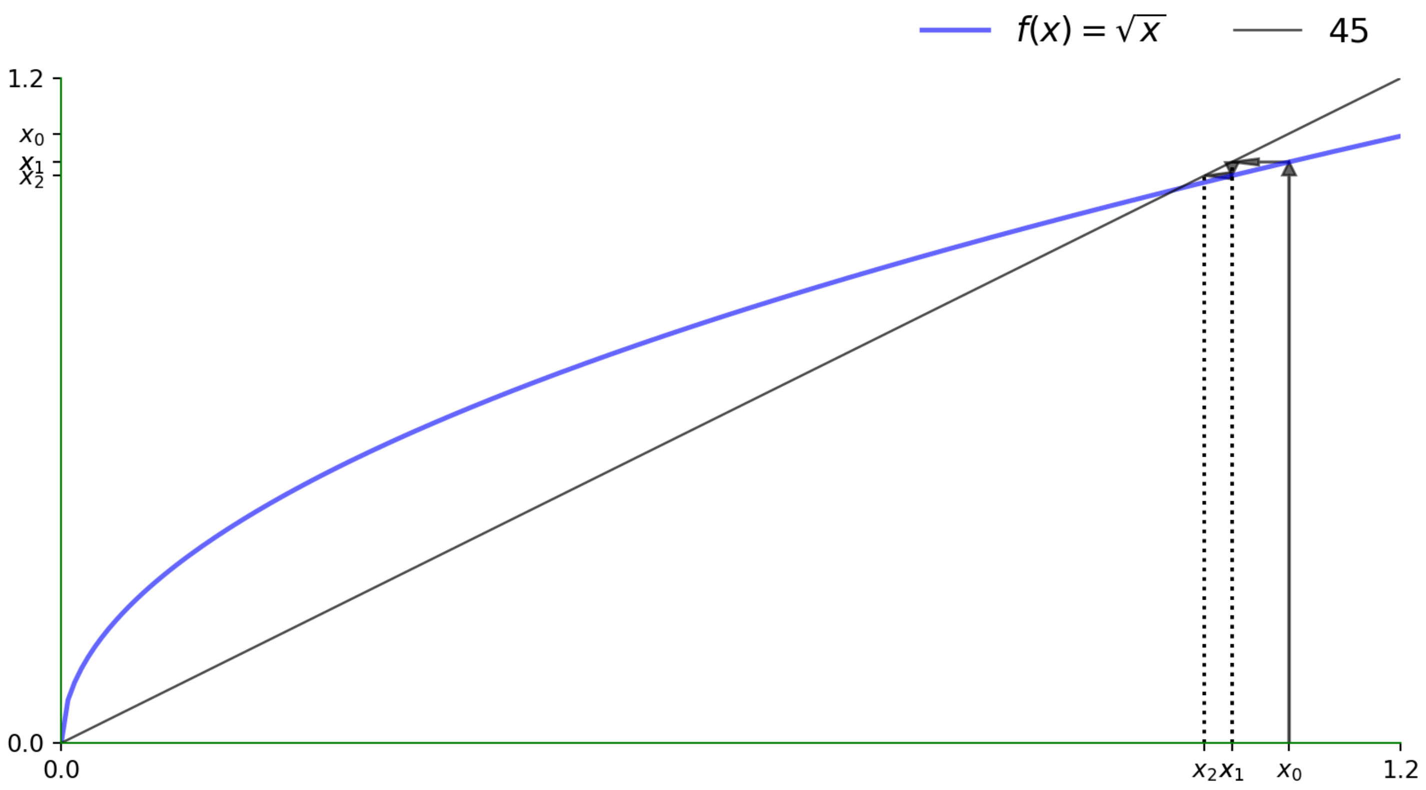
- To use these figures:
  1. Start with any point on the x-axis
  2. Jump to the $f(\cdot)$ for that point to see where it went
  3. Go across to the 45 degree line
  4. Then down to the new value
- Repeat! Useful to interpret dynamics as well as various numerical methods
- Gives intuition on speed of convergence/etc. as well

See **QuantEcon Scalar dynamics** for base code

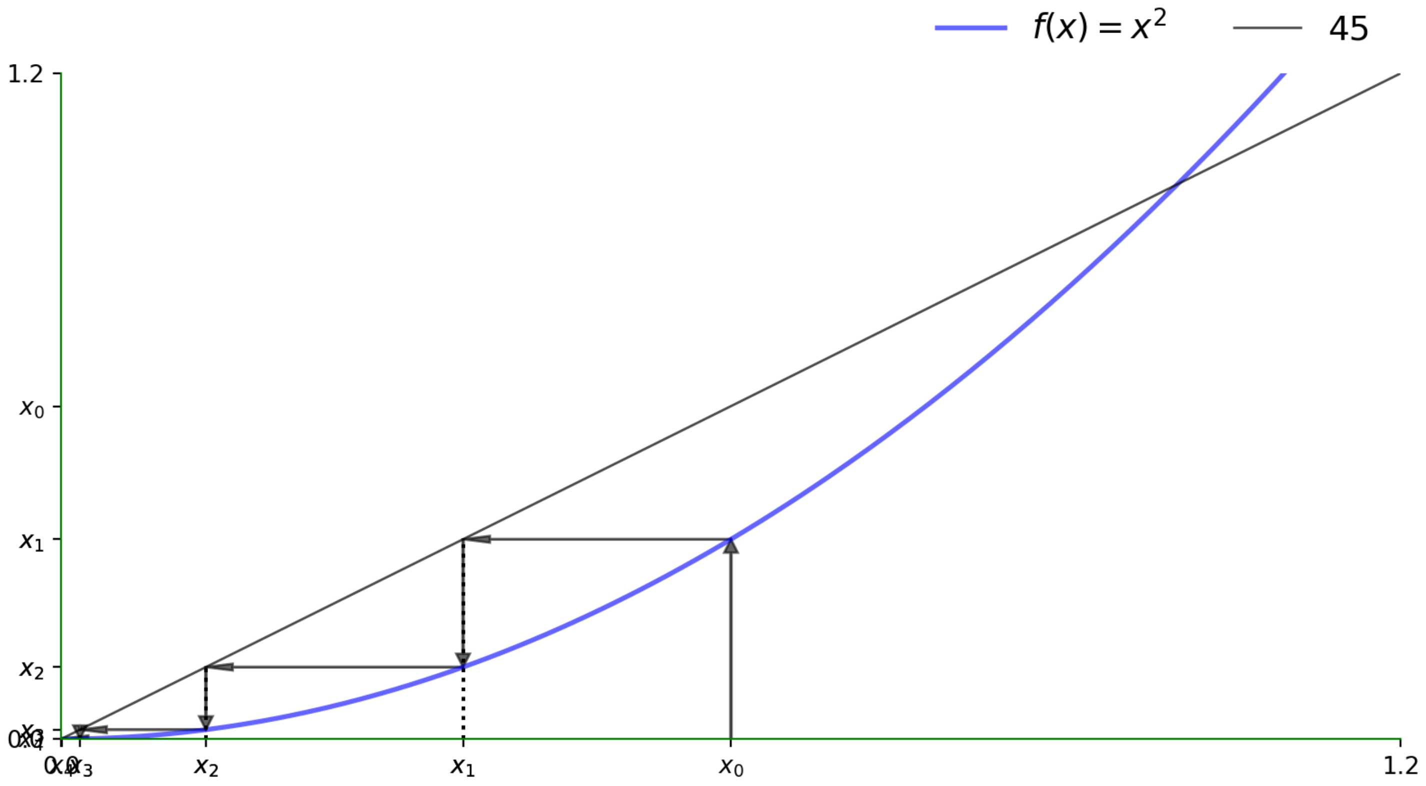
Evaluating the $\sqrt{x}$ near $x = 0.05 > 0$



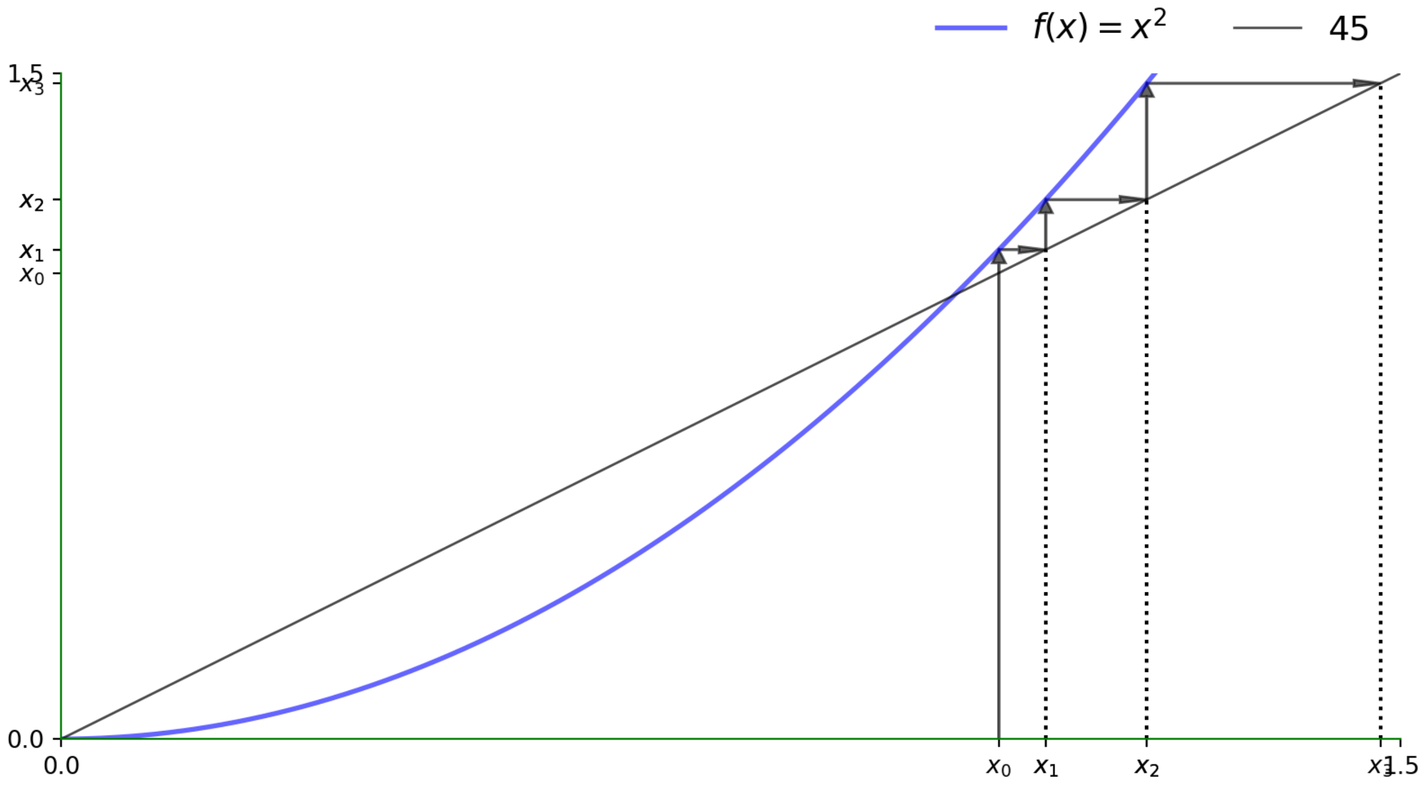
Evaluating the $\sqrt{x}$ near $x = 1.1 > 1$



Evaluating the x^2 for $x = 0.6 < 1$



Evaluating the x^2 for $x = 1.01 > 1$



Linear Dynamics and Stability

Scalar Linear Model

$$x_{t+1} = ax_t + b \equiv f(x_t), \qquad \text{given } x_0$$

$$\begin{aligned}
x_1 &= ax_0 + b \\
x_2 &= ax_1 + b = a^2 x_0 + ab + b \\
&\dots \\
x_t &= a^t x_0 + b \sum_{i=0}^{t-1} a^i = a^t x_0 + b \frac{1 - a^t}{1 - a} \\
x^* &\equiv \lim_{t \to \infty} x_t = \begin{cases} \frac{b}{1-a} & \text{if } |a| < 1 \\ \text{diverges} & \text{if } |a| > 1 \text{ or } a = 1, b \neq 0 \\ \text{indeterminate} & \text{if } a = 1, b = 0 \end{cases}
\end{aligned}$$

Stability and Jacobians

- Given $f(x_t) = ax_t + b$
  - → The Jacobian (derivative since scalar) $\nabla f(x_t) = a$
- Eigenvalues of a scalar are just the value itself, so can write the condition as
  - → Stable at fixed point x^* if $\rho(\nabla f(x^*)) < 1$, where $\rho(A) = \max_i |\lambda_i(A)|$ the spectral radius
  - → Saw this as a condition for stability with higher-dimensional linear systems when looking at Present Discounted Values

Linearization and Stability

- Important condition for stability with nonlinear $f(\cdot)$
- Intuition: assume x^* exists and then
  - → Linearize around the steady state and see if it would be locally explosive
  - → Necessary but not sufficient. $\rho(\nabla f(x^*)) > 1 \implies x^*$ can't be a stable fixed point
- You may see this when working with macro models in Dynare and similar methods in macroeconomics

Linearization

- Assume steady state $x^* = f(x^*)$ exists, with system $x_{t+1} = f(x_t)$
- Take first-order taylor expansion around x^*

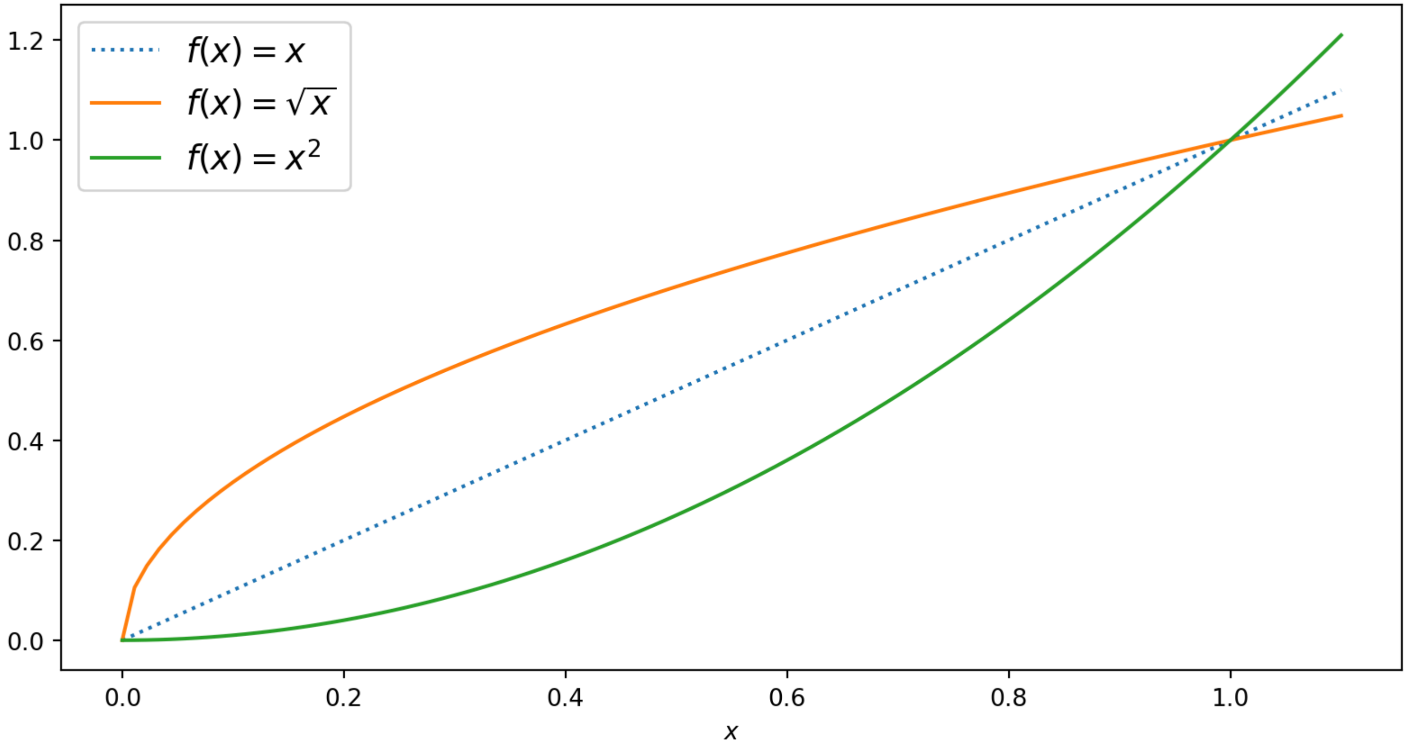
$$
\begin{aligned}
x_{t+1} &= f(x^*) + \nabla f(x^*)(x_t - x^*) + \text{second order and smaller terms} \\
x_{t+1} - x^* &\approx \nabla f(x^*)(x_t - x^*) \\
\hat{x}_{t+1} &\approx \nabla f(x^*)\hat{x}_t
\end{aligned}
$$

- Where the last formulation is common in macroeconomics and time-series econometrics. $\hat{x}_t \equiv x_t - x^*$ is the **deviation from the steady state**
  - → For the linear case, these would all be exact as there are no higher-order terms

Quality of Linearization

- Gives approximate dynamics for a perturbation close to the steady state
  - → May have good approximation far away from x^* if $f(\cdot)$ is close to linear
  - → May have terrible approximations close to x^* if $f(\cdot)$ highly nonlinear/asymmetric
  - → Often log-linearization is used instead, which expresses in percent deviation

Plot Against 45 degree line Reminder



Stability of $\sqrt{x}$ and x^2

- Recall that both had fixed points at $x^* = 0$ and $x^* = 1$
- Lets check derivatives! Let $f_1(x) = \sqrt{x}$ and $f_2(x) = x^2$
  - → $\nabla f_1 x = \frac{1}{2\sqrt{x}}$ and $\nabla f_2(x) = 2x$
- Check spectral radius of the Jacobians (trivial since univariate) at the fixed points:
  - → At $x^* = 0$, $\nabla f_1(0) = \infty$ and $\nabla f_2(0) = 0$
  - → At $x^* = 1$, find $\nabla f_1(1) = \frac{1}{2}$ and $\nabla f_2(1) = 2$
- Interpretation:
  - → $f_1(x)$ is locally explosive at $x^* = 0$ and locally stable at $x^* = 1$
  - → $f_2(x)$ is locally stable at $x^* = 0$ and locally explosive at $x^* = 1$

Solow-Swan Growth Model

Model of Growth and Capital

- An early growth model of economic growth is the **Solow-Swan model**
- Simple model. Details of the derivation for self-study/macro classes:
  - → k_t by capital per worker and y_t is total output per worker
  - → $\alpha \in (0, 1)$ be a parameter which governs the marginal product of capital
  - → $\delta \in (0, 1)$ is the depreciation rate (i.e., fraction of machines breaking each year)
  - → $A > 0$ is a parameter which governs the total factor productivity (TFP)
  - → $s \in (0, 1)$ is the fraction of output used for investment and savings

Capital Dynamics

Then capital dynamics follow a nonlinear difference equation with steady state

$$
\begin{aligned}
y_t &= Ak_t^{\alpha} \\
k_{t+1} &= sy_t + (1-\delta)k_t = sAk_t^{\alpha} + (1-\delta)k_t \equiv g(k_t) \quad \text{given } k_0 \\
k^* &\equiv \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}
\end{aligned}
$$

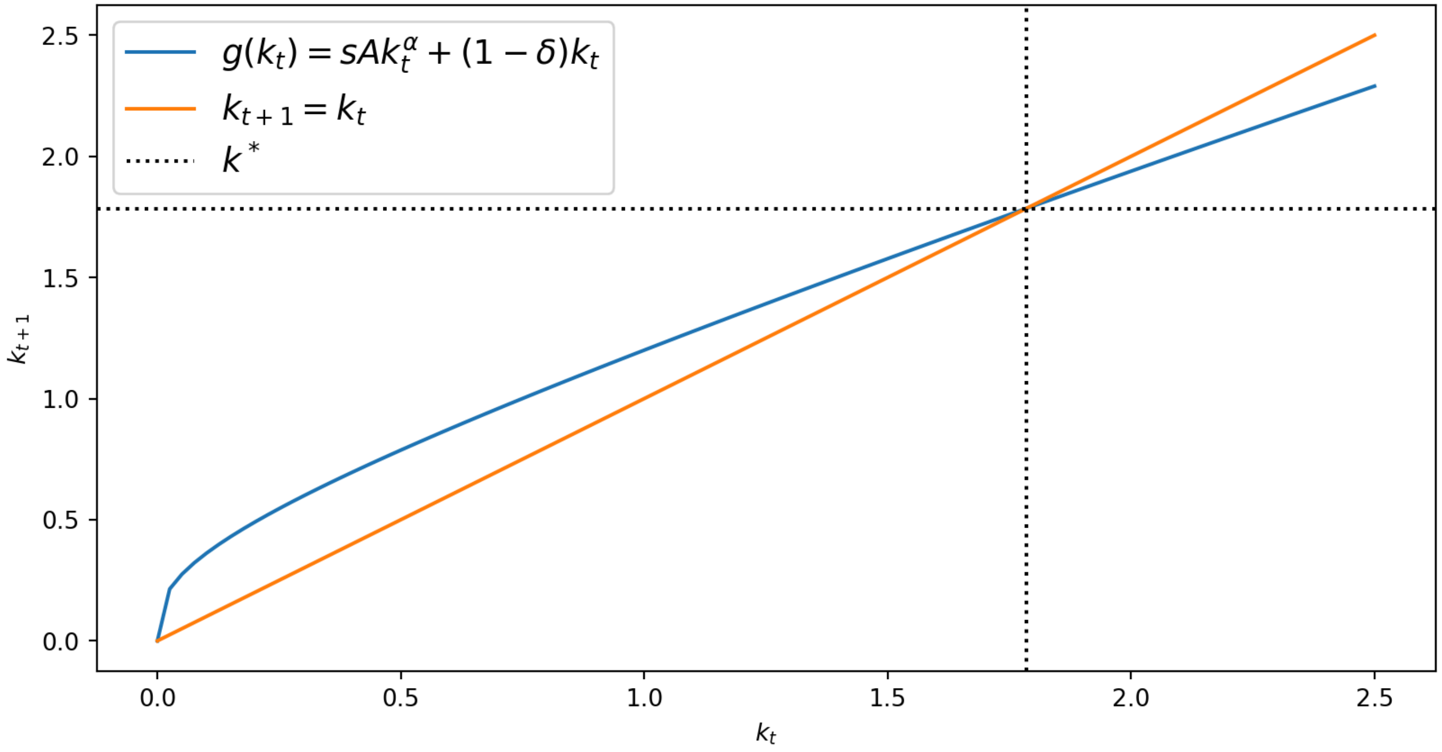
Implementing the Solow-Swan Model

```python
A, s, alpha, delta = 2, 0.3, 0.3, 0.4
def y(k):
    return A*k**alpha
# "closure" binds y, A, s, alpha, delta
def g(k):
    return s*y(k) + (1-delta)*k

k_star = (s*A/delta)**(1/(1-alpha))
k_0 = 0.25
print(f"k_1 = g(k_0) = {g(k_0):.3f},\
k_2 = g(g(k_0)) = {g(g(k_0)):.3f}")
print(f"k_star = {k_star:.3f}")
```

```
k_1 = g(k_0) = 0.546,k_2 = g(g(k_0)) = 0.828
k_star = 1.785
```

Plotting k_t vs. k_{t+1} verifies our k^*



Jacobian of g at the steady state

$$\begin{aligned}\nabla g(k^*) &= \alpha s A k^{*\alpha-1} + 1 - \delta, \qquad \text{substitute for } k^* \\ &= \alpha s A \frac{\delta}{sA} + 1 - \delta = \alpha\delta + 1 - \delta \\ &= 1 - (1-\alpha)\delta < 1\end{aligned}$$

- Key requirements were $\alpha \in (0,1)$ and $\delta \in (0,1)$
- The spectral radius of a scalar is just that value itself.
- The spectral radius of $||\nabla g(k^*)|| < 1$, a necessary condition for k^* stable
- **Aside:** macroeconomics, industrial organization, etc. this is related to contraction mappings and Blackwell's condition

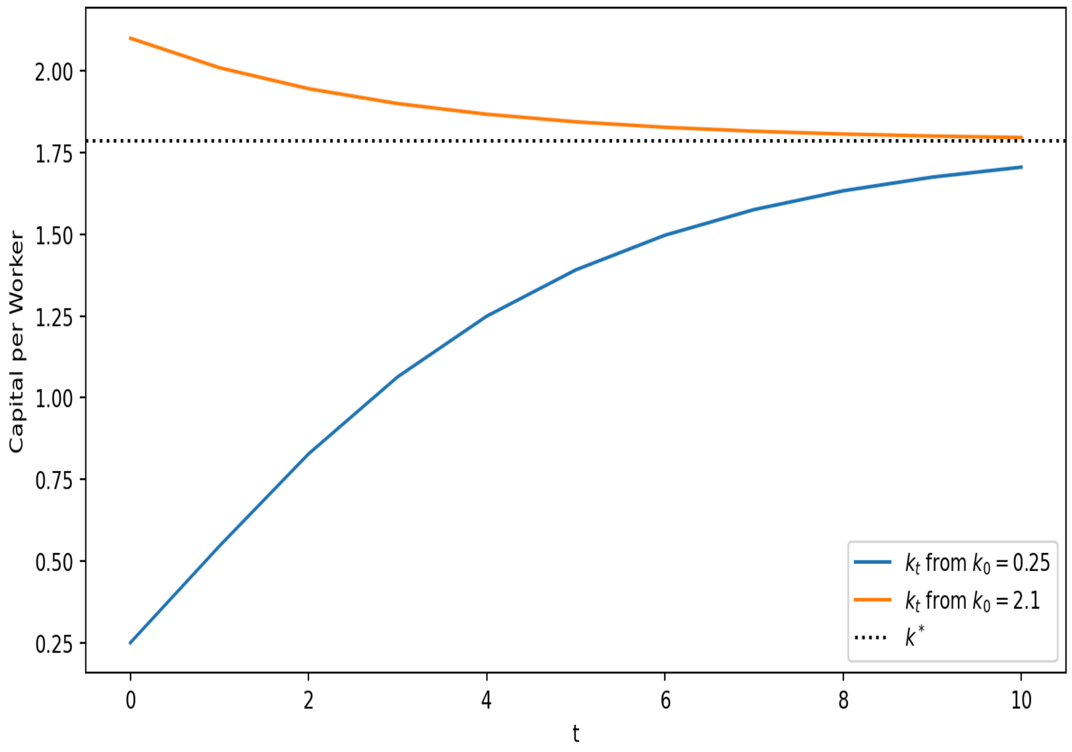
Simulation

```
# Generic function, takes in a function!
def simulate(f, X_0, T):
    X = np.zeros((1, T+1))
    X[:,0] = X_0
    for t in range(T):
        X[:,t+1] = f(X[:,t])
    return X
T = 10
X_0 = np.array([0.25]) # initial condition
X = simulate(g, X_0, T) # use with our g
print(f"X_{T} = {X[:,T]}")
```

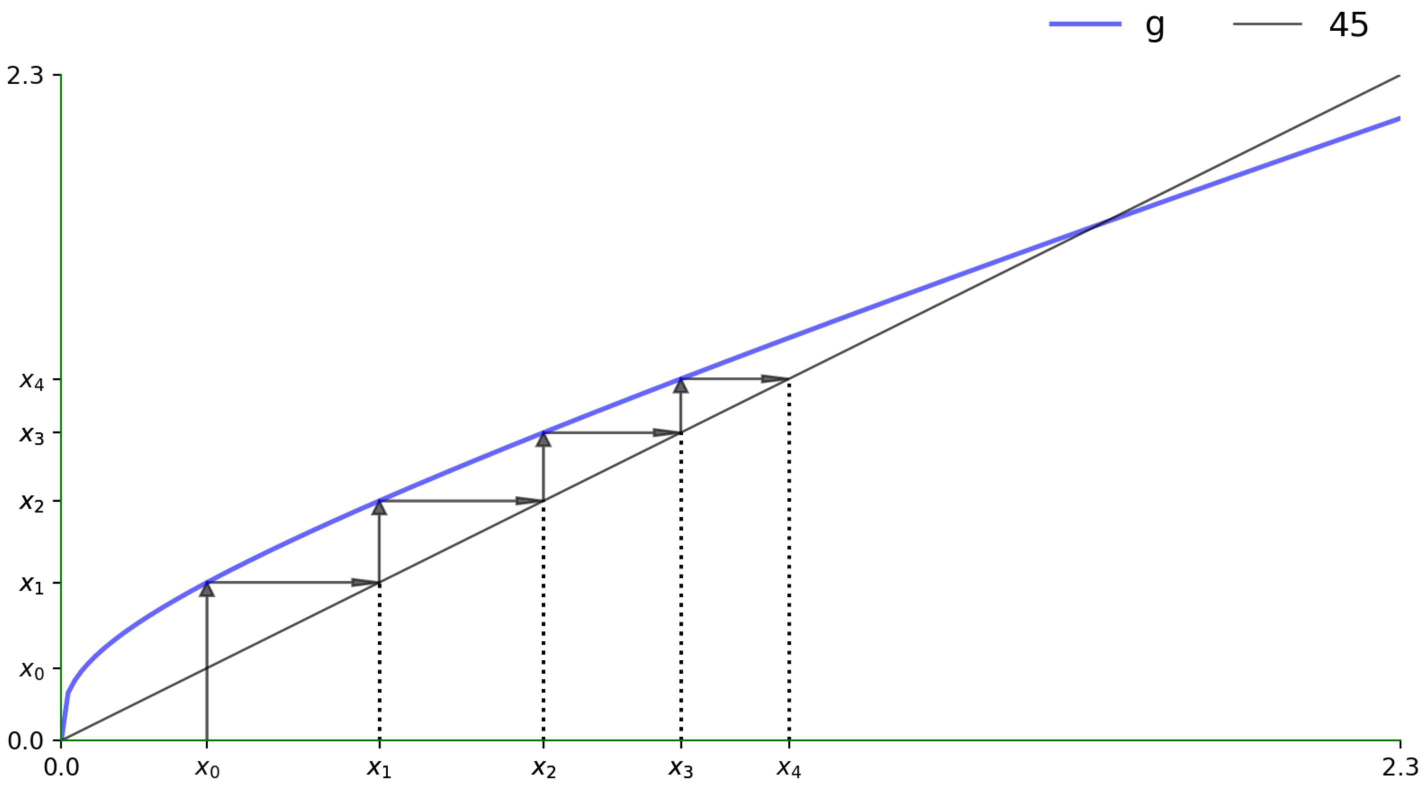
```
X_10 = [1.70531835]
```

Capital Transition from $k_0 < k^*$ and $k_0 > k^*$

```python
X_1 = simulate(g, X_0, T) # use with our g
X_2 = simulate(g, np.array([2.1]) , T)
fig, ax = plt.subplots()
ax.plot(range(T+1), X_1.T,
  label=r"$k_t$ from $k_0 = 0.25$")
ax.plot(range(T+1), X_2.T,
  label=r"$k_t$ from $k_0 = 2.1$")
ax.set(xlabel="t", ylabel="Capital per Worker")
ax.axhline(y=k_star, linestyle=':',
  color='black',label=r"$k^*$")
ax.legend()
plt.show()
```



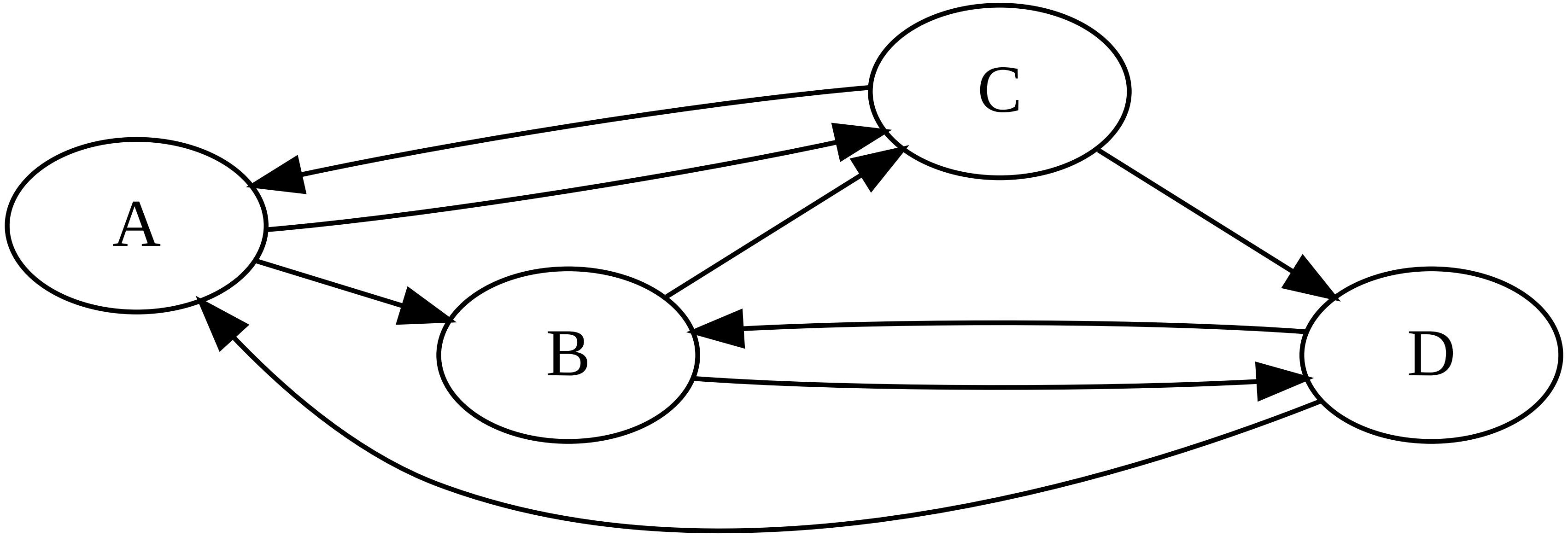
Trajectories Using the 45 degree Line



PageRank and Other Applications

Network of Web Pages

- Consider A, B, C, D as a set of web pages with links given below



Create an Adjacency Matrix

- We can summarize the network of web pages with 1 or 0 if there is a link between two pages. Pages won't link to themselves
- This is in (arbitrary) order: A, B, C, D

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

PageRank Algorithm

One interpretation of this is that you can

- Start on some page
- With equal probability click on all pages linked at that page
- Keep doing this process and then determine what fraction of time you spend on each page

Probabilistic Interpretation

Alternatively,

- Start with a probability distribution, r_t that you will be on any given page (i.e. $r_{nt} \geq 0$ and $\sum_{n=1}^{4} r_{nt} = 1$)
- Iterate the process to see the probability distribution after you click the next links
- Repeat until the probability distribution doesn't change.

Adjacency Matrix to Probabilities

- To implement, we want to put the same probability on going to any link for a given page (i.e. each row)

$$S = \begin{pmatrix} 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & 0 \end{pmatrix}$$

Probabilities Evolution

- Now, we can see what happens after we click on a page
- For a given $\boldsymbol{r}_t$ distribution of probabilities across page, I can see the new probabilities distribution as

$$r_{t+1} = S^\top r_t$$

- Motivation to learn more probability and Markov Chains (next set of lectures)

Fixed Points and Eigenvectors

- What is a fixed point of this process?
- Eigenvector of $S^\top$ associated with $\lambda = 1$ eigenvalue!
- The real PageRank is a little more subtle (adds in dampening) but the same basic idea
- Learn numerical algebra to use in practice. It is infeasible to actually compute the eigenvector of a huge matrix with a decomposition.