ECON526: Problem Set 2

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Name: ____________________

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Instructions

- Ensure you modify the field above with your **name and student number above immediately**
- Modify directly and do not rename the file as Canvas will automatically append your name to it.
- Submit both the `.ipynb` and an exported `html` version of this file.
- See here for instructions.
- Do not change the filenames. Canvas automatically appends your name/student number to submitted files.

Setup

Use the following packages and imports

```
import numpy as np
import matplotlib.pyplot as plt
import scipy
from numpy.linalg import cond, matrix_rank, norm
from scipy.linalg import inv, solve, det, eig, lu, eigvals
from scipy.linalg import solve_triangular, eigvalsh, cholesky
```

Question 1

Question 1.1

Generate a random matrix $A \in \mathbb{R}^{10\times 10}$ and random vector $b \in \mathbb{R}^{10}$ of uniformly distributed floating points between 0 and 1. Hint: use `np.random.rand(10, 10)`

and solve $Ax = b$ as a linear system with

1. `scipy.linalg.solve`
2. `scipy.linalg.inv`

Modify

```
N = 10
# modify here
```

What matrix decomposition would `scipy.linalg.solve` likely use in this case?

Answer:

(double click to edit your answer)

Question 1.2

The product of any matrix with its transpose is symmetric. Prove it. Hint: the definition of symmetric matrix B is if $B = B^T$. Use this to take the transpose of $B \equiv A^T A$.

Answer:

(double click to edit your answer)

Question 1.3

Using your matrix A from before, construct the symmetric $B = A^T A$.

Verify it is symmetric, and then find out if it is positive definite. Hint: you will need to use `eigvals` or `eigs`.

```
# modify here, using A from above
```

Question 1.4

Now solve the system $Bx = b$ using `solve` and `inv`. If the matrix was shown to be symmetric or positive definite before, then use that in your solution

```
# modify here
```

Question 2

Question 2.1

Take the matrix $A \in \mathbb{R}^{100\times 5}$

Check if it is full rank

```
# modify here
N = 100
K = 5
A = np.random.rand(N, K)
```

Question 2.2

Take that previous matrix in Q2.1 and append a new column to it, so that it is now $\hat{A} \in \mathbb{R}^{100\times 6}$ such that the matrix will still have a rank of 5 and not 6. Hint: lots of ways to append a vector to a matrix in numpy, including `numpy.column_stack` and `numpy.concatenate`

```
# modify here
```

Question 2.3

Take the A and the $\hat{A}$ from before, and form $B = AA^T$ and $\hat{B} = \hat{A}\hat{A}^T$. What are the ranks?

```
# modify here
```

Could we do a cholesky decomposition of this matrix? Check and/or explain why not if you can't

Answer:

(double click to edit your answer)

Question 3

Question 3.1

Take the following $B \in \mathbb{R}^{N \times N}$ symmetric matrix and do an eigendecomposition (spectral decomposition in this case since symmetric), and print out the eigenvalues

```
N = 10
A = 2.0 * np.random.rand(N, N)
B = A.T @ A

# modify here
# Lambda, Q = ....
```

Question 3.2

For your matrix above, calculate its spectral radius

```
# modify here
```

Question 4

Question 4.1

Take the vector $\hat{x}_1 \in \mathbb{R}^2$

```
x_hat_1 = np.array([1, 2])
```

Verify that it is not a unit length vector (i.e. $\|\hat{x}_1\| \neq 1$) then create a new x_1 that is a unit length vector in the same direction as $\hat{x}_1$ (i.e. $||x_1|| = 1$)

```
# modify here
# x_1 = ...
```

Question 4.2

Now find a x_2 which is also a unit length vector, but is orthogonal to x_1. Check it with `np.dot(x_1, x_2)` approx 0 and `norm(x_2)` approx 1. Hint: many ways to do this by hand in $\mathbb{R}^2$ and fulfill the requirements, such as simple rotations.

```
# modify here
# x_2 = ...
```

Question 4.3

The vectors x_1 and x_2 are now an orthonormal set. Form the matrix $Q = \begin{bmatrix} x_1 & | & x_2 \end{bmatrix}$ and verify the condition for orthonormality (i.e. $Q^T Q = I \implies Q^{-1} = Q^T$)

```
# Q = np.column_stack((x_1, x_2))
# modify here
```

Question 4.4

Create a matrix A such that: 1. Q from the previous question are its eigenvectors 2. The spectral radius of A is 1.0 3. A is positive definite.

Hint: create a matrix of eigenvalues Λ and then do an eigendecomposition in reverse

```
# modify here
```