High Dimensional Probability and Concentration of Measure

Machine Learning Fundamentals for Economists

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Overview

Summary

- High-dimensional spaces exhibit counterintuitive geometric and probabilistic properties
- Key phenomena:
  - → **Curse of dimensionality**: most volume is in corners, uniform sampling fails
  - → **Concentration of measure**: functions of many variables are "essentially constant"
  - → **Johnson-Lindenstrauss**: random projections preserve distances
  - → **Near-orthogonality**: random vectors are almost perpendicular
  - → **Random algorithms**: leverage concentration to provide probabilistic error guarantees
- These concepts are foundational for understanding why ML methods work
- Applications to solving structural economic models covered in subsequent lecture

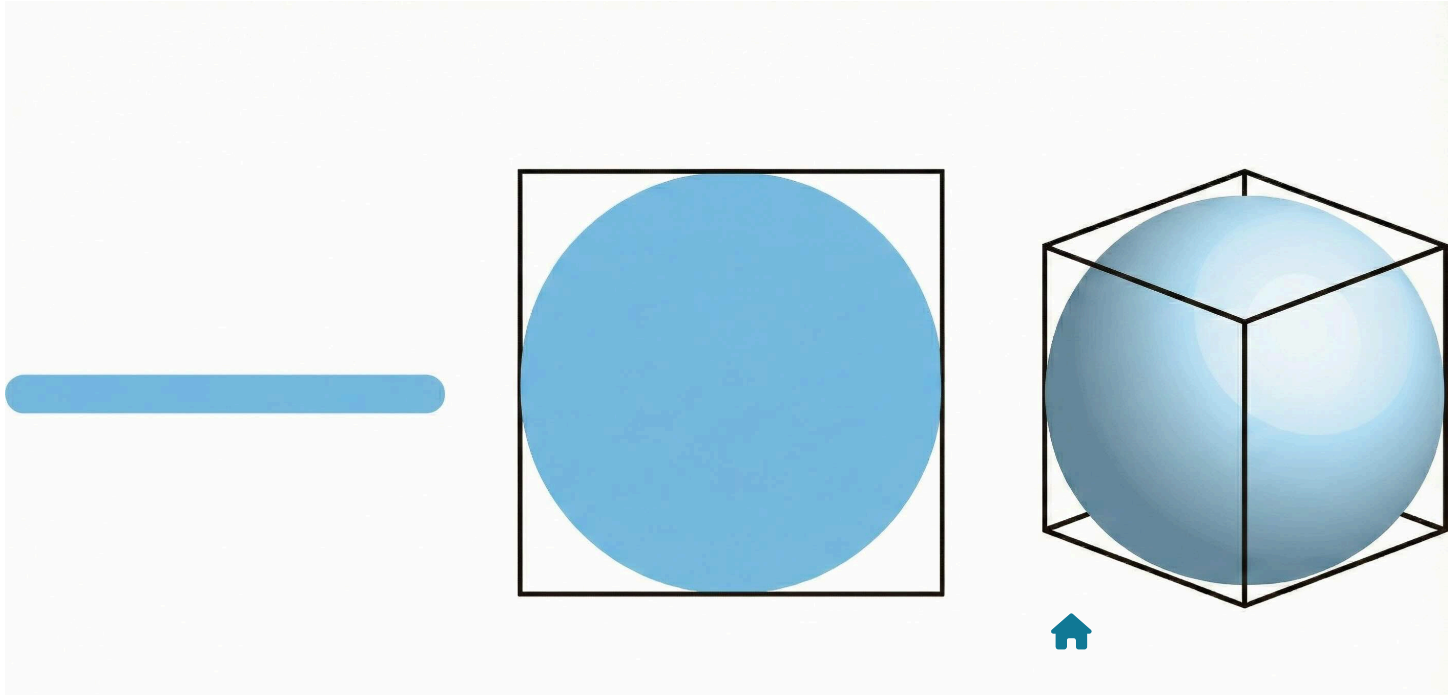
References

- Primary reference: Vershynin (2018)
- Concentration inequalities: Boucheron, Lugosi, and Massart (2013)
- Classic reference: Ledoux (2001)

High-Dimensional Geometry

Hypercubes Are All Corners

- Counter-intuitive geometry: **in high-dimensions, hypercubes are all corners**
- Consider volume within a 1/2 radius of the origin in a hypercube $[-1/2, 1/2]^N$
  - → Hypersphere inscribed in hypercube, touching all “faces” for all N
  - → $V(1) = 1$. The origin is close to every point on the line segment.
  - → $V(2) = \pi(1/2)^2, V(3) = \frac{4}{3}\pi(1/2)^3, V(10) \approx 0.002, V(50) = 1.536743 \times 10^{-28}$



A High-Dimensional Space is a Lonely Place (Bernhard Schölkopf)

- Any X (e.g., steady-state) becomes increasingly distant from other regions of $\mathcal{X}$
- Approximating functions with a grid or sampled on uniform $[X_{\min}, X_{\max}]^N$ distribution
  - → Almost all of the volume in high-dimensional spaces is where you do not need it
  - → i.e., not only does the num grid points, M, need to increase exponentially with N, but asymptotically **almost every point is wasted** for many applications
  - → Calculating numerical expectations has similar issues (see Betancourt (2018))
- How could these possibly generalize from $\mathcal{D}$ to $\mathcal{X}$? Is it really so hopeless?
  - → ML and deep learning methods succeed in approximating functions on even richer spaces (e.g., images, videos, and NLP). Same with Bayesian methods like HMC.
  - → Krusell-Smith approximations work very well in practice for many problems

High Dimensional Probability

Minimizing Uniform vs. Population Risk

- **Uniform bounds on Errors**: impossible in general without exponential cost in N

$$\min_{f \in \mathcal{F}} \sup_{(x,y) \in \mathcal{X}} \left[(f(x) - y)^2 \right]$$

- **Statistical learning**: minimizes regions weighted by a distribution μ^*

$$\min_{f \in \mathcal{F}} \mathbb{E}_{(x,y) \sim \mu^*} \left[(f(x) - y)^2 \right]$$

  → Helps if probability distribution is "concentrated" in specific regions of $\mathcal{X}$

  → Doesn't help doesn't if μ^* is itself "all corners" (e.g., uniformly distributed)

  → Challenging if "worst case" matters (e.g., game with unconstrained best-responses)

Concentration of Measure Phenomenon

"A random variable that depends in a Lipschitz way on many independent variables (but not too much on any of them) is essentially constant." (Ledoux (2001))

- i.e., functions of high-dimensional random variables have small variance if
  - → The correlation between individual random variables are controlled
  - → Lipshitz-like: If no single coordinate dominates, the function "averages out"
- **Non-asymptotic bounds** that improve with dimension (i.e., larger N is better, not worse)
- While geometry in $\mathcal{X}$ may be subject to a curse of dimensionality, functions of random variables on that space can become increasingly predictable

Example: Norm of Isotropic Gaussian

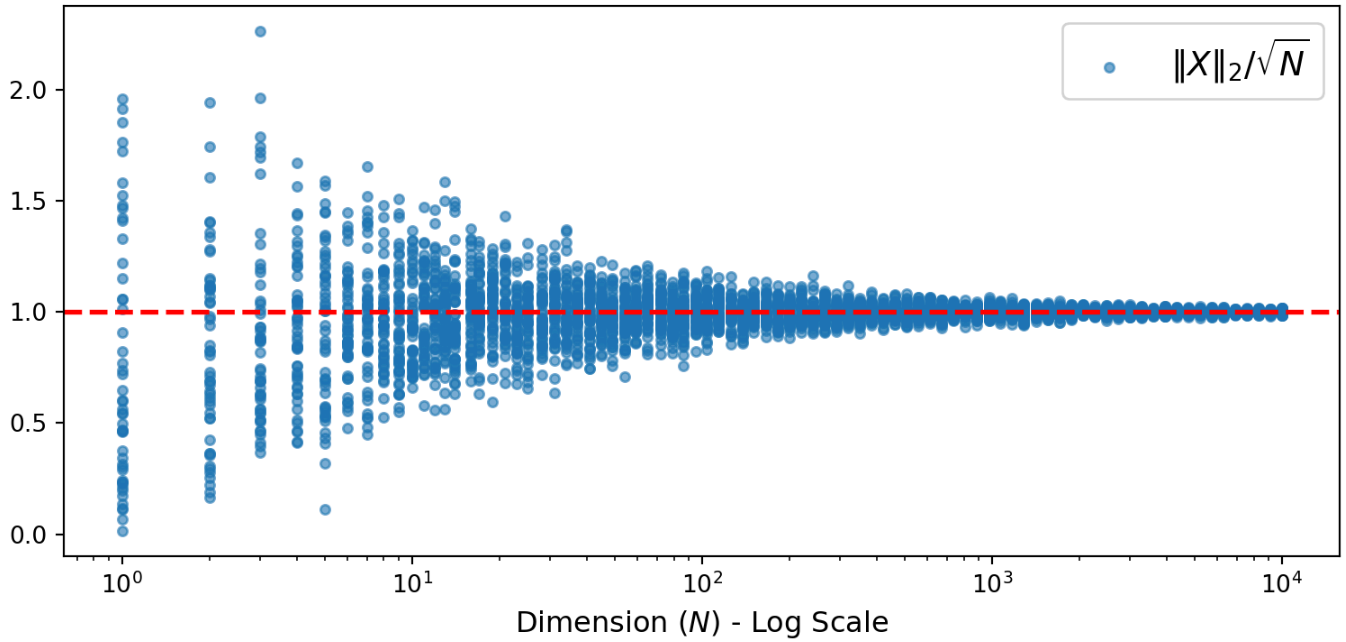
- Let $X \sim \mathcal{N}(0_N, I_N)$ be an N-dimensional standard Gaussian
- Consider $f(X) = \|X\|_2/\sqrt{N}$
- Can prove the concentration bound:

$$\mathbb{P}\left(|f(X) - 1| \geq \epsilon\right) \leq \exp(-c\epsilon^2 N)$$

for some constant $c > 0$ independent of N.

- The ratio $\|X\|_2/\sqrt{N}$ becomes increasingly predictable as N grows
- Convergence is **exponentially fast** in N. Blessing, not curse, of dimensionality!

Visualization of Concentration of Norms



- 50 draws of $X \sim \mathcal{N}(0_N, I_N)$ for each dimension N
- As N increases, all samples concentrate tightly around 1

The Soap Bubble Effect

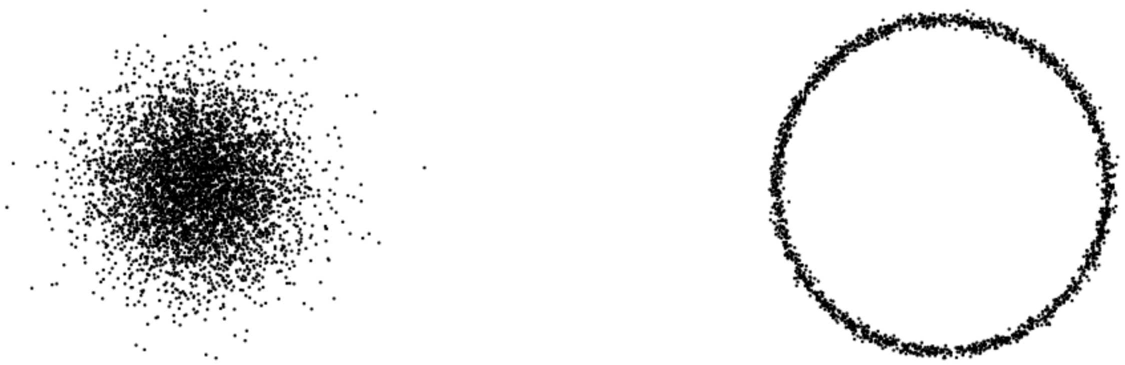


Figure 3.5 Sampling from the standard normal distribution in 2D (left) and its heuristic visualization in high dimensions (right). In high dimensions, the standard normal distribution closely resembles the uniform distribution on a sphere of radius $\sqrt{n}$.

From Vershynin (2018): 2D Gaussian samples vs. random 2D projections of high-dimensional samples

- 2D Gaussians are distributed around the origin with an intuitive **peak**
- High-dim Gaussians are distributed roughly uniformly on a **spherical shell** of radius $\sqrt{N}$
- Points are nowhere near the mode/mean/median (i.e., $X = 0_N$)

Lipschitz Concentration (Informal)

Definition: Lipschitz Function

A function $f : \mathbb{R}^N \to \mathbb{R}$ is L-**Lipschitz** if for all $x, y \in \mathbb{R}^N$:

$$|f(x) - f(y)| \leq L\|x - y\|_2$$

- Lipschitz functions cannot change too fast
- For $f(x) = \|x\|_2$, we have $L = 1$
- Key insight: Lipschitz functions of Gaussians concentrate around their mean
- Can be extended to non-Lipschitz functions bounding expected gradients, etc.

Gaussian Concentration Inequality

Proposition: Concentration for Lipschitz Functions of Gaussians

Let $X \sim \mathcal{N}(0_N, I_N)$ and let $f : \mathbb{R}^N \to \mathbb{R}$ be L-Lipschitz. Then:

$$\mathbb{P}\left(\left|f(X) - \mathbb{E}[f(X)]\right| \geq t\right) \leq 2\exp\left(-\frac{t^2}{2L^2}\right)$$

- The bound is **independent of dimension N**!
- For normalized functions (gradient $O(1/\sqrt{N})$), concentration improves with N
  - → i.e., approximate average output of an industry vs. total output of an industry
- Many extensions: bounded differences, sub-Gaussian tails, etc. (Boucheron, Lugosi, and Massart (2013))

When Concentration Fails

Breaking Concentration: Three Mechanisms

The Gaussian concentration inequality requires:

1. **Isotropic covariance**: $X \sim \mathcal{N}(0, I_N)$
2. **Lipschitz function**: bounded gradient $||\nabla f|| \leq L$
3. **Distributed dependence**: function doesn't rely too heavily on any single coordinate

Violating (1) or (2) can **break concentration**, while violating (3) **prevents improvement** with dimension.

Operator Norm and Spectral Radius

Definition: Operator Norm and Spectral Radius

The **operator norm** (spectral norm) of $A \in \mathbb{R}^{N \times N}$:

$$||A||_{\text{op}} = \sup_{x \neq 0} \frac{||Ax||_2}{||x||_2}$$

The **spectral radius** of A:

$$\varrho(A) = \max_i |\lambda_i(A)|$$

- In general: $\varrho(A) \leq ||A||_{\text{op}}$
- For **symmetric PSD** matrices (e.g., covariance Σ): $||\Sigma||_{\text{op}} = \varrho(\Sigma) = \lambda_{\max}(\Sigma)$
- Concentration bounds depend on $\varrho(\Sigma)$: larger spectral radius → weaker concentration

Non-Isotropic Gaussians: The Setup

- Consider $Z \sim \mathcal{N}(0, \Sigma)$ with general covariance Σ
- Write $Z = \Sigma^{1/2} X$ where $X \sim \mathcal{N}(0, I_N)$
- For an L-Lipschitz function f, define $g(X) = f(\Sigma^{1/2} X)$

$$|g(x) - g(y)| = \left|f(\Sigma^{1/2}x) - f(\Sigma^{1/2}y)\right| \leq L\|\Sigma^{1/2}(x-y)\|_2$$

- Using operator norm: $\|\Sigma^{1/2}(x-y)\|_2 \leq \left\|\Sigma^{1/2}\right\|_{\mathrm{op}} \|x-y\|_2 = \sqrt{\varrho(\Sigma)}\|x-y\|_2$

Concentration with Non-Isotropic Gaussian

Proposition: Non-Isotropic Gaussian Concentration

For $Z \sim \mathcal{N}(0, \Sigma)$ and $f : \mathbb{R}^N \to \mathbb{R}$ that is L-Lipschitz:

$$\mathbb{P}\left(|f(Z) - \mathbb{E}[f(Z)]| \geq t\right) \leq 2\exp\left(-\frac{t^2}{2L^2\varrho(\Sigma)}\right)$$

- The **effective Lipschitz constant** is $L\sqrt{\varrho(\Sigma)}$
- Concentration depends on the **spectral radius** of the covariance
- If $\varrho(\Sigma)$ grows with N, concentration can **degrade** with dimension

Perfect Correlation Breaks Concentration

- Consider **equicorrelation**: $\Sigma = \sigma^2 \left((1-\rho)I_N + \rho \mathbf{1}_N \mathbf{1}_N^\top\right)$
  - → Diagonal: $\Sigma_{ii} = \sigma^2$ (each variable has variance σ^2)
  - → Off-diagonal: $\Sigma_{ij} = \sigma^2 \rho$ (correlation ρ between all pairs)
- **Eigenvalues**:
  - → $\sigma^2(1-\rho)$ with multiplicity $N-1$
  - → $\sigma^2(1+(N-1)\rho)$ with multiplicity 1
- As $\rho \to 1$: $\varrho(\Sigma) \to \sigma^2 N$

Perfect Correlation: The Bound Degrades

- With $\varrho(\Sigma) = \sigma^2 N$, the concentration bound becomes:

$$\mathbb{P}\left(\left|f(Z) - \mathbb{E}[f(Z)]\right| \geq t\right) \leq 2\exp\left(-\frac{t^2}{2L^2\varrho(\Sigma)}\right) = 2\exp\left(-\frac{t^2}{2L^2\sigma^2 N}\right)$$

- **Concentration gets worse with dimension!**
- **Intuition**: With perfect correlation, all N variables move together
  - → Effectively have only **one degree of freedom**, not N
  - → No "averaging out" across independent sources of variation
- This is why independence (or weak dependence) is crucial for concentration

Functions of Single Coordinates

- Consider $f(X) = X_1$ with $X \sim \mathcal{N}(0, I_N)$
- Gradient: $\nabla f = (1, 0, \dots, 0)^\top$, so $L = 1$
- Concentration bound:

$$\mathbb{P}\left(|X_1| \geq t\right) \leq 2e^{-t^2/2}$$

- **No improvement with dimension!** The bound is the same for $N = 10$ or $N = 10{,}000$
- $X_1 \sim \mathcal{N}(0, 1)$ regardless of N—adding more coordinates doesn't help

Contrast: Sample Mean Concentrates

- Compare with $\bar{X} = \frac{1}{N} \sum_{i=1}^{N} X_i$
- Gradient: $\nabla \bar{X} = \frac{1}{N}(1, 1, \dots, 1)^\top$
- Lipschitz constant: $L = \|\nabla \bar{X}\|_2 = \frac{1}{\sqrt{N}}$
- Concentration bound:

$$\mathbb{P}\left(|\bar{X}| \geq t\right) \leq 2 \exp\left(-\frac{t^2 N}{2}\right)$$

- **Improves with dimension!** Exponentially tighter as N grows
- $\bar{X} \sim \mathcal{N}(0, 1/N)$: variance shrinks as $1/N$

Key Insight: Coordinate Dependence

When Does Concentration Improve with N?

Concentration improves with dimension when **no single coordinate dominates**:

- $\partial f / \partial X_i = O(1/\sqrt{N})$ for all i → concentration improves
- $\partial f / \partial X_1 = O(1)$, others ≈ 0 → no improvement

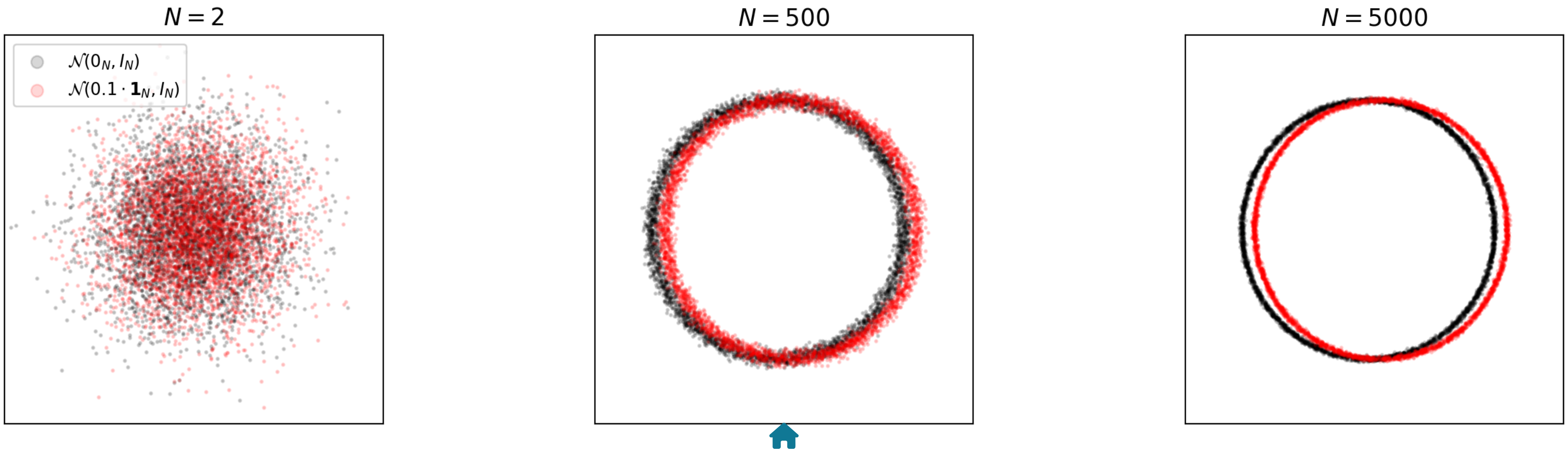
- **Intuition**: The function must “average out” across many coordinates
- Functions that depend on all coordinates roughly equally benefit from high dimensions
- Functions concentrated on few coordinates behave like low-dimensional problems

Concentration is a Double-Edged Sword

Shifted Mean: Increasing Separation

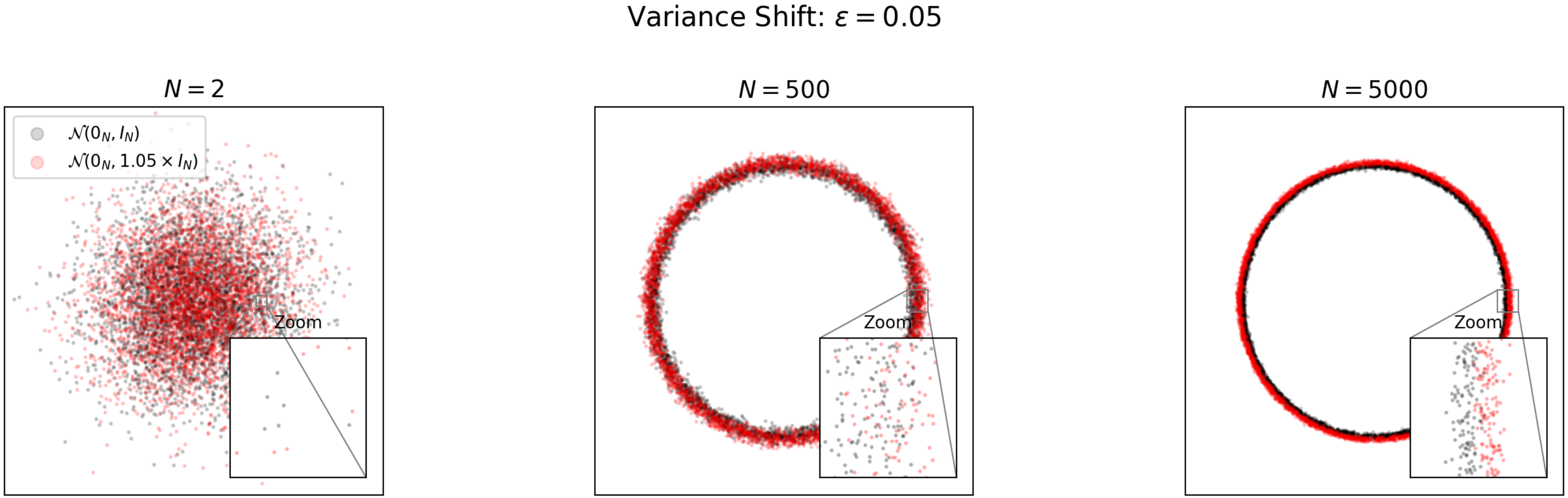
- Consider two distributions: $\mathcal{N}(0_N, I_N)$ vs. $\mathcal{N}(\epsilon \cdot \mathbf{1}_N, I_N)$
- The Euclidean distance between means is $\epsilon\sqrt{N}$
- As N grows, distributions separate completely despite small per-coordinate shift
  - → Showing random 2D projections from 4000 sampled points for visualization

Mean Shift: $\varepsilon = 0.1$



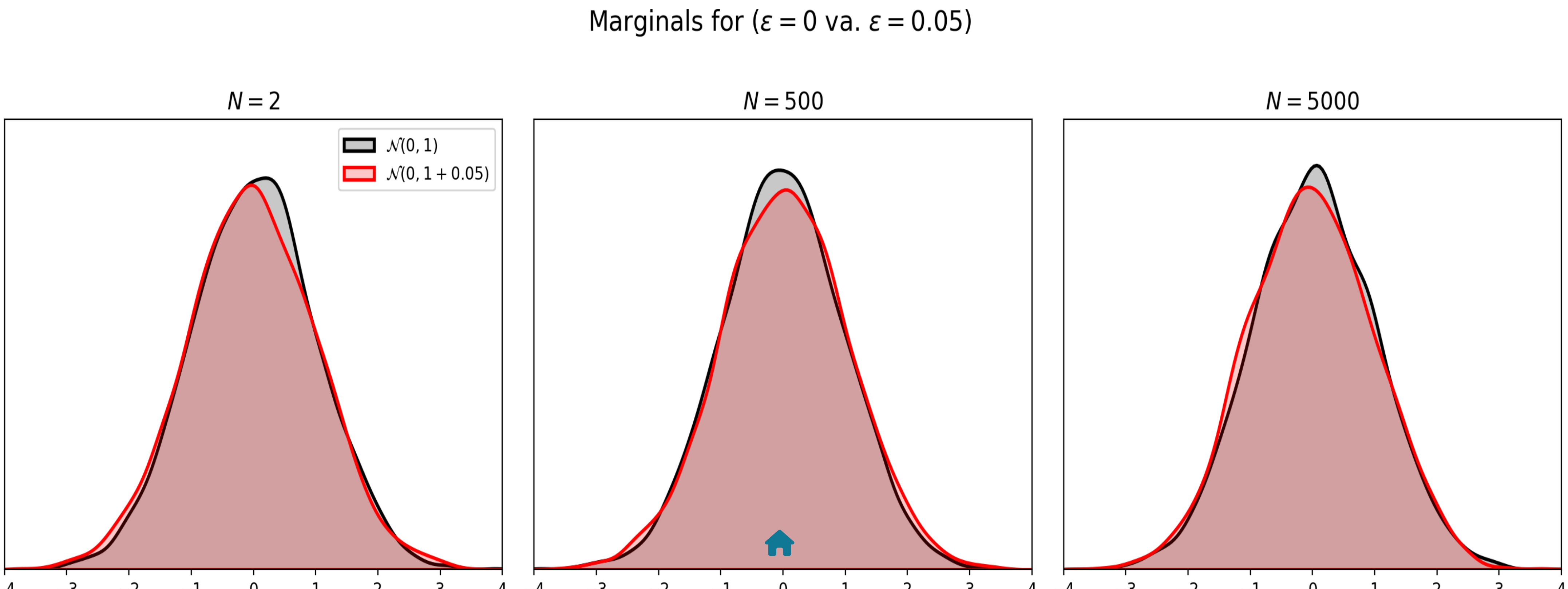
Variance Shift: Dangerous and Difficult to Detect

- Consider: $\mathcal{N}(0_N, I_N)$ vs. $\mathcal{N}(0_N, (1+\epsilon)I_N)$
- A 5% variance increase causes complete separation in high dimensions
  - → Sampling more points from a slightly misspecified distribution may make statistical learning worse, not better.



Marginals Are Deceptive

- With the shifted variance, the marginal distribution of any single coordinate is **similar** for both distributions
  - → Showing the empirical 1D marginals of x_1 from 4000 samples
- Recall that the joint distributions are completely separated in high dimensions



Key Insight: Double-Edged Sword

- **Within a distribution**: Concentration is helpful
  - → Functions become predictable
  - → Few samples may suffice for accurate estimation
- **Across distributions**: Concentration is dangerous
  - → Small changes in distribution parameters cause complete separation
  - → A model trained on one distribution may fail entirely on a slightly different one
- **Implication for learning**:
  - → Must ensure training distribution $\mathcal{D} \sim \mu^*$ matches test distribution
  - → "Out of distribution" generalization is fundamentally hard and requires changes to the training algorithm.
  - → Representation learning becomes especially important. e.g., if functions were summarized well-approximated by mean and variance they could be learned and generalize well.

Random Algorithms in High Dimensions

Motivation

- High-dimensional economic models often involve objects that are:
  - → Too large to store explicitly (e.g., large design or transition matrices)
  - → Too costly to evaluate deterministically (e.g., integrals over $\mathbb{R}^{100}$ or OLS with two-way fixed effects with millions of observations)
- **Random algorithms** use carefully designed randomness to:
  - → Control approximation error probabilistically
  - → Replace exponential complexity with polynomial or logarithmic complexity
- These are not heuristics—they satisfy **non-asymptotic error bounds**:
  - → $\mathbb{P}(|\mathrm{Error}| > \epsilon) \leq \delta$ where δ includes functions of the number of samples, lipshitz constants, etc. Often decreasing, rather than increasing, in dimension N
  - → Justified by **concentration of measure**

Johnson–Lindenstrauss as a Guiding Principle

Proposition: Johnson–Lindenstrauss (JL) Lemma

For any $\epsilon \in (0,1)$, $\delta \in (0,1)$, and points $\{x_1, \dots, x_M\} \subset \mathbb{R}^N$, let $R \in \mathbb{R}^{k \times N}$ be a random matrix (e.g., Gaussian or Rademacher entries) scaled by $1/\sqrt{k}$. If

$$k = \mathrm{O}\left(\epsilon^{-2} \log(M^2/\delta)\right),$$

then with probability at least $1 - \delta$, for all i, j:

$$(1 - \epsilon)\|x_i - x_j\|_2^2 \;\le\; \|Rx_i - Rx_j\|_2^2 \;\le\; (1 + \epsilon)\|x_i - x_j\|_2^2.$$

- **Key insight**: Randomness can preserve structure (i.e., geometry as captured by norms/similarity) **uniformly** over many objects
- JL provides a paradigm for probabilistic error control in random algorithms
- Other random algorithms share the same concentration foundation, but operate in different regimes of estimating functionals vs. preserving geometry

Why Does k Depend on M, Not N?

- The projection dimension k depends on **how many distances** to preserve, not original dimension
- **With 2 points**: only 1 distance to preserve
  - → Can project to $k = \mathrm{O}(\epsilon^{-2})$ dimensions
  - → A single projected distance concentrates around the true distance
- **With M points**: must preserve $\binom{M}{2} = \mathrm{O}(M^2)$ distances **simultaneously**
  - → Need $k = \mathrm{O}(\epsilon^{-2} \log(M^2/\delta))$ for all to hold with failure probability at most δ
  - → The $\log M$ factor comes from a union bound over all pairs

Why Random Projections Work: The Math

- Each individual distance concentrates:

$$\mathbb{P}\left(\left|\|Rx\|_2^2 - \|x\|_2^2\right| > \epsilon\|x\|_2^2\right) \lesssim e^{-c\epsilon^2 k}$$

- **Union bound** over $\mathrm{O}(M^2)$ pairs: multiply failure probability by M^2
- To keep total failure $\leq \delta$: need $e^{-c\epsilon^2 k} \cdot M^2 \leq \delta$
- Solving: $k \gtrsim \epsilon^{-2}(2\log M + \log(1/\delta)) = \epsilon^{-2}\log(M^2/\delta)$
- **Dimension N appears nowhere!** Randomness “averages out” across all N coordinates

JL Preserves Fixed Sets, Not New Points

Important Limitation

JL preserves distances **among a fixed set** of points $\{x_1, \dots, x_M\}$.

It provides no guarantee for distances involving a **new point** x_{M+1}.

- A new point would require either:
  - → A fresh random projection, or
  - → Adding it to the set and increasing k accordingly
- **Implication**: JL is ideal for **static datasets**, less so for streaming/online settings
- For adaptively chosen points, different techniques are needed (e.g., oblivious subspace embeddings)

A Taxonomy of Concentration-Based Random Algorithms

Algorithm type	What is preserved	Economic application
JL / Random Projections	Pairwise geometry / inner products	High-dimensional moment inequalities
Hutchinson Trace	Spectral trace / quadratic forms	Variance decomposition
Randomized SVD / Sketching	Low-rank structure	Factor models, macro-finance
Stochastic Trace Estimation	Operator traces	Continuous-time asset pricing

We now look at canonical economic applications.

Application: Kline, Saggio, and Sølvsten (2020)

- **Goal**: Decompose wage variance into worker and firm components
- **Variance component**: For coefficient vector β, estimate $\theta = \beta^\top A \beta$
  - → Matrix A selects which component (e.g., worker effects, firm effects, covariance)
- **Key matrices**:
  - → Projection matrix: $P = X(X^\top X)^{-1} X^\top$ (leverages P_{ii} for leave-out corrections)
  - → Bias correction: $B = (X^\top X)^{-1} A$ (combines design with component selector)
- **Challenge**: P and B are massive - cannot form explicitly

Kline, Saggio, and Sølvsten (2020): Why Traces?

- The naive plug-in $\hat{\theta} = \hat{\beta}^\top A \hat{\beta}$ is biased:

$$\mathbb{E}[\hat{\beta}^\top A \hat{\beta}] = \beta^\top A \beta + \sigma^2 \cdot \mathbf{Tr}(B)$$

- To debias: estimate $\mathbf{Tr}(B)$ via Hutchinson's trick with Rademacher vectors $z_j \in \{\pm 1\}^N$:

$$\mathbf{Tr}(B) = \mathbb{E}_z[z^\top B z] \approx \frac{1}{m} \sum_{j=1}^{m} z_j^\top (B z_j)$$

→ Only need one linear solve of $(X^\top X)(Bz) = Az$ for Bz per draw z.

→ No explicit matrix required. Use preconditioned iterative solvers for high-dimensional design matrices (see iterative methods for more)

Application: Continuous-Time Asset Pricing and Stochastic Trace

- Consider a high-dimensional diffusion on state $X_t \in \mathbb{R}^N$:

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_t \in \mathbb{R}^N, \quad dW_t \in \mathbb{R}^K \quad \text{Brownian motion}$$

  → $\mu : \mathbb{R}^N \to \mathbb{R}^N$ is the drift and $\sigma : \mathbb{R}^N \to \mathbb{R}^{N\times K}$ is the volatility

- The infinitesimal generator, $\mathscr{A}$, for this process is

$$\mathscr{A}f(X) = \mu(X)^\top \nabla f(X) + \frac{1}{2}\mathrm{Tr}\left(\sigma(X)\sigma(X)^\top \nabla^2 f(X)\right)$$

Hamilton-Jacobi-Bellman Equation (HJBE)

Simple HJBE for asset pricing,

$$\rho V(X) = u(X) + \mathcal{A}V(X) = u(X) + \mu(X)^{\top}\nabla V(X) + \frac{1}{2}\mathrm{Tr}\left(\sigma(X)\sigma(X)^{\top}\nabla^2 V(X)\right),$$

- Discount rate $\rho > 0$
- Utility function $u : \mathbb{R}^N \to \mathbb{R}$
- Subject to initial and boundary conditions/transversality

Random Algorithm for Diffusion Term

- **Goal**: Compute $\mathbf{Tr}\left(\sigma(X)\sigma(X)^\top\nabla^2 V(X)\right)$
  - → Direct computation is $\mathbf{O}(N^2)$ - infeasible for large N
  - → Use cyclic trace trick to get $\mathbf{Tr}\left(\sigma(X)\sigma(X)^\top\nabla^2 V(X)\right) = \mathbf{Tr}\left(\sigma(X)^\top\nabla^2 V(X)\,\sigma(X)\right)$
- **Hutchinson estimator**: For $z \sim \mathcal{N}(0, I_K)$, let $w = \sigma(X)z \in \mathbb{R}^N$

$$\mathbf{Tr}\left(\sigma(X)^\top\nabla^2 V(X)\,\sigma(X)\right) = \mathbb{E}_{w\sim\mathcal{N}(0,\sigma(X)\sigma(X)^\top)}[w^\top\nabla^2 V(X)\,w]$$

- Estimate with m samples (each with $\mathbf{O}(N)$ cost):

$$\mathbf{Tr}\left(\sigma(X)^\top\nabla^2 V(X)\,\sigma(X)\right) \approx \frac{1}{m}\sum_{j=1}^{m} w_j \cdot \left(\nabla^2 V(X)\cdot w_j\right), \quad w_j \sim \mathcal{N}(0, \sigma(X)\sigma(X)^\top)$$

  - → $\nabla^2 V(X)\cdot w$ is a **Hessian-Vector Product (hvp)** - use AD such as JAX's hvp

Application (IN PROGRESS): Chiong and Shum (2019)

- **Goal**: Estimate discrete-choice models with extremely high-dimensional regressors $X^t \in \mathbb{R}^d$
- Use random-projections to reduce dimension from d to $k \ll d$
- Estimation relies on **cyclic monotonicity** of choice probabilities
- **Cyclic Monotonicity Inequalities**: For any cycle $t_1, \dots, t_L$:

$$\sum_{\ell=1}^{L} \left(X^{t_{\ell+1}}\beta - X^{t_\ell}\beta\right)^\top p^{t_\ell} \leq 0.$$

This yields a finite collection of **inner-product inequalities** in β.

Random Projection in Chiong and Shum (2019)

Apply a random projection $\widetilde{X}^t = RX^t$, with $k \ll d$.

- The same estimator is run on projected data
- JL-type bounds ensure that the inner products defining the inequalities are preserved with high probability
- The feasibility region of the estimator is approximately unchanged

Note

Essence: One random projection must preserve *many* inequalities simultaneously — this is exactly the JL regime.

References

Betancourt, Michael. 2018. "Probabilistic Computation." https://betanalpha.github.io/assets/case_studies/probabilistic_computation.html.

Boucheron, Stéphane, Gábor Lugosi, and Pascal Massart. 2013. *Concentration Inequalities: A Nonasymptotic Theory of Independence*. Oxford University Press.

Chiong, Khai Xiang, and Matthew Shum. 2019. "Random Projection Estimation of Discrete-Choice Models with Large Choice Sets." *Management Science* 65 (1): 256–71. https://doi.org/10.1287/mnsc.2017.2928.

Kline, Patrick, Raffaele Saggio, and Mikkel Sølvsten. 2020. "Leave-Out Estimation of Variance Components." *Econometrica* 88 (5): 1859–98. https://doi.org/10.3982/ECTA16410.

Ledoux, Michel. 2001. *The Concentration of Measure Phenomenon*. 89. American Mathematical Society.

Vershynin, Roman. 2018. *High-Dimensional Probability: An Introduction with Applications in Data Science*. Vol. 47. Cambridge university press.